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Bivariate Pareto–Feller Distribution Based on Appell Hypergeometric Function

Christian Caamaño-Carrillo [1], Moreno Bevilacqua [2,3], Michael Zamudio-Monserratt [4] and Javier E. Contreras-Reyes [5,*]

1 Departamento de Estadística, Facultad de Ciencias, Universidad del Bío-Bío, Concepción 4081112, Chile; chcaaman@ubiobio.cl
2 Facultad de Ingeniería y Ciencia, Universidad Adolfo Ibañez, Viña del Mar 2580335, Chile; moreno.bevilacqua@uai.cl
3 Dipartimento di Scienze Ambientali, Informatica e Statistica, Ca' Foscari University of Venice, 30172 Venice, Italy
4 NCT/Odisseia-Observatory of Socio-Environmental Dynamics (UnB), Universidade de Brasilia, Brasilia 70910-900, Brazil; michael.zamudio@postgrado.uv.cl
5 Instituto de Matemática, Física y Estadística, Facultad de Ingeniería y Negocios, Universidad de Las Américas, Sede Viña del Mar, 7 Norte 1348, Viña del Mar 2531098, Chile
* Correspondence: jecontrr@uc.cl; Tel.: +56-9-7960-8218

Abstract: The Pareto–Feller distribution has been widely used across various disciplines to model "heavy-tailed" phenomena, where extreme events such as high incomes or large losses are of interest. In this paper, we present a new bivariate distribution based on the Appell hypergeometric function with marginal Pareto–Feller distributions obtained from two independent gamma random variables. The proposed distribution has the beta prime marginal distributions as special case, which were obtained using a Kibble-type bivariate gamma distribution, and the stochastic representation was obtained by the quotient of a scale mixture of two gamma random variables. This result can be viewed as a generalization of the standard bivariate beta I (or inverted bivariate beta distribution). Moreover, the obtained bivariate density is based on two confluent hypergeometric functions. Then, we derive the probability distribution function, the cumulative distribution function, the moment-generating function, the characteristic function, the approximated differential entropy, and the approximated mutual information index. Based on numerical examples, the exact and approximated expressions are shown.

Keywords: generalized gamma distribution; beta prime marginal distributions; generalized hypergeometric function; moment generation function; entropy

MSC: 60E05; 62H10; 33C65

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1. Introduction

The Pareto distribution has been a key tool in economics and for modeling wealth distribution, as well as in other areas like insurance and finance, where capturing extreme events is crucial [1]. However, real-world data often exhibit more complex structures than those described by the classical Pareto distribution [2]. In this context, Feller (1971) [3] proposed an extension called the Pareto–Feller distribution which includes an additional shape parameter, allowing the distribution to better fit a wider range of phenomena with thicker or heavy tails as needed [1].

The Pareto–Feller distribution has been widely used across various disciplines to model "heavy-tailed" phenomena, where extreme events such as high incomes or large losses are of interest. The Pareto–Feller distribution emerged from the need for a distribution that offers greater flexibility in data modeling by introducing an additional parameter to control the tail shape and skewness, thus providing a more accurate description of empirical data compared with the standard Pareto distribution [2]. This distribution has found applications in various fields such as risk theory, river flow modeling, and natural disaster analysis due to its ability to represent both heavy tails and asymmetric distributions. The additional flexibility provided by this distribution is especially valuable in situations where conventional distributions, like the classical Pareto distribution, fail to capture the observed variability and extreme behavior in the data. Notable variants include Pareto type I, used in wealth analysis [1], the generalized Pareto distribution for modeling extreme events [4], and the Lomax or Pareto type II distribution, which is applied in survival analysis [5]. Other important variants are Pareto type IV, which offers greater flexibility in tail shapes [6], and the truncated Pareto distribution, which is used in scenarios with physical upper limits [7]. Additionally, the log-logistic distribution, which is employed in contexts similar to the Pareto distribution but with greater flexibility in the tails, is used in survival analysis and system failure studies [8].

The Pareto–Feller distribution can be constructed as a location-scale transformation of the ratio of two independent gamma-distributed random variables. This method allows the distribution to capture a wide range of tail behaviors and offers flexibility in modeling heavy-tailed phenomena. The use of gamma distributions for generating such models is well documented in the statistical literature. Specifically, more flexible models can be obtained through an appropriate transformation of a bivariate gamma distribution or independent copies of it. This approach is appealing because the correlation of the transformed bivariate gamma distribution is directly tied to the correlation of the original bivariate gamma distribution. Such constructions are commonly used in spatial data modeling. Examples include $t-$distributed spatial models [9], Weibull spatial models [10], and Poisson spatial models [11]. On the other hand, Kotz et al. (2004) [12] described similar methods for transforming distributions via ratios of gamma variables which are widely applicable in reliability and survival analysis contexts [12].

Since the construction of the Pareto–Feller distribution is related to the gamma distribution, it is necessary to first define the bivariate gamma distribution to develop the bivariate case of the Pareto–Feller distribution. Thus, we start by defining a sequence of independent normal random variables and show how these lead to a gamma distribution as follows. Let Z_{ik}, $i = 1, \dots, \nu$ with $\nu > 2$ be a sequence of independent standardized normal random variables whose correlation function is given by $Corr(Z_{ik}, Z_{jk}) = \rho$, $i \neq j$, and let

$$W_k = \sum_{i=1}^{\nu} \frac{Z_{ik}^2}{\alpha}, \quad \alpha > 0, \quad k = 1, 2. \tag{1}$$

Then, W_k is a random variable with a gamma marginal distribution (i.e., $W_k \sim Gamma(\nu/2, \alpha/2)$, where $\nu/2$ represents the shape parameter and $\alpha/2$ represents the rate parameter), with the probability density function (pdf) being given by

$$f_{W_k} = \frac{\alpha^{\nu/2}}{2^{\nu/2}\Gamma(\nu/2)} w_k^{\nu/2-1} e^{-\alpha w_k/2},$$

where $\mathbb{E}[W_k] = \nu/\alpha$ and $Var(W_k) = 2\nu/\alpha^2$ [10].

The construction of a bivariate Pareto–Feller distribution is derived from the ratio of two bivariate gamma distributions as shown in Section 2. We consider the bivariate vector $\mathbf{W} = (W_1, W_2)^\top$, where the stochastic representation of W_k, $k = 1, 2$ is given in Equation (1). Vere-Jones [13] showed that the distribution of $\mathbf{W}$ has a correlated bivariate gamma distribution with the parameters $\nu > 0$ and $\gamma > 0$, while the pdf is given by

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{2^{-\nu}\gamma^{\nu}(w_1 w_2)^{\nu/2-1} e^{-\frac{\gamma(w_1+w_2)}{2(1-\rho^2)}}}{\Gamma\left(\frac{\nu}{2}\right)(1-\rho^2)^{\nu/2}} \left(\frac{\gamma\sqrt{\rho^2 w_1 w_2}}{2(1-\rho^2)}\right)^{1-\nu/2} I_{\nu/2-1}\left(\frac{\gamma\sqrt{\rho^2 w_1 w_2}}{(1-\rho^2)}\right), \tag{2}$$

where $I_\alpha(\cdot)$ is the usual modified Bessel function of the γ-order of the first kind. Gamma variables can be used as building blocks for the construction of flexible non-Gaussian variables. Henceforth, we will call $\mathbf{W}$ a gamma random vector with an underlying correlation ρ [14,15] such that the correlation of the gamma bivariate is $\rho_{\mathbf{W}} = \rho^2$. Moreover, when

$\rho = 0$, Equation (2) can be written as the product of two independent gamma random variables (i.e., $W_k \sim Gamma(\nu/2, \alpha/2)$, $k = 1, 2$). Thus, zero pairwise correlation implies pairwise independence, as in the Gaussian case. The pdf $f_{\mathbf{W}}$ was first discussed in [16], and its properties were studied in [17,18].

The bivariate Pareto–Feller distribution represents a less commonly discussed extension in the statistical literature. It builds upon the principles established by the univariate Pareto distribution. The authors of [19] addressed extreme value distributions and included discussions which may be relevant to bivariate generalizations. Additionally, the authors of [20] provided further insights into bivariate distributions, enriching the understanding of Pareto–Feller distributions. These references offer both theoretical and practical frameworks for researching and applying Pareto–Feller distributions in bivariate contexts. The latter works motivated this paper, which presented a bivariate Pareto–Feller distribution built from an Appell hypergeometric function.

This paper is organized as follows. Section 2 presents the bivariate Pareto–Feller distribution. In particular, the pdf, cumulative distribution function (cdf), joint moment-generating function (mgf), characteristic functions, cross-product moment function, mean, variance, covariance, and correlation function are presented. In Section 3, some approximations of the differential entropy and, consequently, the mutual information index are presented. Finally, some discussions and conclusions are presented in Section 4. All simulations included special hypergeometric functions such as the Appell hypergeometric one, and all were implemented in `R` 4.4.1. software [21] using the `zipfR` and `hypergeo` packages. All proofs of theorems and propositions can be found in the Appendix A.

2. Bivariate Pareto–Feller Distribution

Let us define a random variable V with support on the positive real line, defined as a scale mixture of two gamma random variables:

$$V = \frac{W}{R}, \tag{3}$$

where $W \sim Gamma(\alpha/2, 1)$, $\alpha > 0$ and $R \sim Gamma(\nu/2, 1)$, $\nu > 0$. Then, V is a random variable with a marginal distribution of the beta I type or beta prime [12,22] and denoted by $V \sim Be'(\nu/2, \alpha/2)$, with the pdf given by

$$f_V(v) = \frac{\Gamma\left(\frac{\nu+\alpha}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} v^{\nu/2-1}(v+1)^{-(\nu+\alpha)/2}.$$

The beta prime distribution is anchored to a shape parameter of the gamma distributions. This construction was previously proposed in [23–25].

Based on the stochastic representation in Equation (3), we consider the bivariate vector of $\mathbf{V} = (V_1, V_2)^\top$, where

$$V_i = \frac{W_i}{R_i}, \quad i = 1, 2; \tag{4}$$

Here, $\mathbf{W} = (W_1, W_2)^\top$ and $\mathbf{R} = (R_1, R_2)^\top$ are two correlated bivariate gamma distributions with correlations $\rho_W = \rho_R = \rho^2$, where $W_i \sim Gamma(\alpha/2, 1)$, $\alpha > 0$ and $R_i \sim Gamma(\nu/2, 1)$, $\nu > 0$, $R_i \perp W_j$, $\forall i, j$, $i, j = 1, 2$. Thus, $V_i \sim Be'(\nu/2, \alpha/2)$, $i = 1, 2$.

A new bivariate distribution with a beta prime marginal distribution obtained from the Kibble-type bivariate gamma distribution given in Equation (2) is presented in the following theorem. This result can be viewed as a generalization of the standard bivariate beta I distribution (or inverted bivariate beta distribution) [26].

Theorem 1. *Let W and R be two independent gamma random variables, and let $V = WR^{-1}$. Then, the pdf of $\mathbf{V} = (V_1, V_2)^\top \in \mathbb{R}_+ \times \mathbb{R}_+$ is given by*

$$f_{\mathbf{V}}(\mathbf{v}) = \frac{(v_1v_2)^{\nu/2-1}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)[(v_1+1)(v_2+1)]^{-(\nu+\alpha)/2}}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \times F_4\left(\frac{\nu+\alpha}{2},\frac{\nu+\alpha}{2};\frac{\nu}{2},\frac{\alpha}{2};\frac{\rho^2 v_1v_2}{(v_1+1)(v_2+1)},\frac{\rho^2}{(v_1+1)(v_2+1)}\right), \tag{5}$$

where F_4 is an Appell hypergeometric function of the fourth kind, defined as

$$F_4(a,b;c,c';w,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{k+m}(b)_{k+m}w^kz^m}{k!m!(c)_k(c')_m}, \quad |\sqrt{w}|+|\sqrt{z}|<1.$$

The special functions F_4 and the Gaussian hypergeometric function ${}_2F_1$ are related through the identity

$$F_4(a,b;c,c';w,z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_kz^k}{k!(c')_k}{}_2F_1(a+k,b+k;c;w), \quad |\sqrt{w}|+|\sqrt{z}|<1.$$

where $(a)_k := \Gamma(a+k)/\Gamma(a)$, for which $k \in \mathbb{N}\cup\{0\}$ is the Pochhammer symbol. The Gaussian hypergeometric function is a special case of more general power series, where the generalized hypergeometric function is defined for $p,q = 0,1,2,\dots$ as

$${}_pF_q(a_1,a_2,\dots,a_p;b_1,b_2,\dots,b_q;x) := \sum_{k=0}^{\infty}\frac{(a_1)_k,(a_2)_k,\dots,(a_p)_k}{(b_1)_k,(b_2)_k,\dots,(b_q)_k}\frac{x^k}{k!}$$

When $\rho = 0$, the pdf in Theorem 1 involves the product of two independent beta prime random variables $Be' \sim (\nu/2,\alpha/2)$.

We now consider a new random variable Y, defined as

$$Y := \mu + \frac{q}{c}V^{1/p}, \tag{6}$$

where $Y \geq \mu \geq 0$, $c = \frac{\Gamma\left(\frac{\nu}{2}+\frac{1}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{1}{p}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}$, $q > 0$, and $p > 0$. This random variable is a marginal Pareto–Feller distribution. Specifically, using the notation of [20], we marginally have $Y \sim PF(\mu,q/c,1/p,\nu/2,\alpha/2)$ with a density defined by

$$f_Y(y) = \frac{\frac{(cp)}{q}\left(\frac{c(y-\mu)}{q}\right)^{p\nu/2-1}\Gamma\left(\frac{\nu+\alpha}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\left[\left(\frac{c(y-\mu)}{q}\right)^p+1\right]^{-(\nu+\alpha)/2}, \quad y > \mu,$$

and a mean and variance given by

$$\mathbb{E}[Y] = \mu + q,$$
$$Var(Y) = q^2\left[\frac{\Gamma\left(\frac{\nu}{2}+\frac{2}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{2}{p}\right)}{c\Gamma\left(\frac{\nu}{2}+\frac{1}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{1}{p}\right)}-1\right],$$

respectively, with $\alpha p > 4$ [3].

The Pareto–Feller distribution includes as special cases different types of Pareto random variables definitions (type I, II, III, and IV; see [20]) and the so-called beta prime one. If we consider $\mu = 0$, then $Y \sim PF(0,q/c,1/p,\nu/2,\alpha/2)$, and $\mathbf{Y} = (Y_1,Y_2)^\top$ is a random vector with a marginal Pareto–Feller distribution. A new bivariate distribution based on marginal Pareto–Feller distributions is presented in the following theorem.

Theorem 2. *Let $\mathbf{Y} = (Y_1,Y_2)^\top$, where $Y_i := \frac{q_i}{c}V_i^{1/p}$, $i = 1,2$. The pdf of $\mathbf{Y}$ is given by*

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{\frac{(cp)^2}{q_1q_2}\left(\frac{c^2y_1y_2}{q_1q_2}\right)^{p\nu/2-1}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\left[\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q2}\right)^p+1\right)\right]^{-(\nu+\alpha)/2}\times$$
$$F_4\left(\frac{\nu+\alpha}{2},\frac{\nu+\alpha}{2};\frac{\nu}{2},\frac{\alpha}{2};\frac{\rho^2(c^2y_1y_2)^p(q_1q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)},\frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right). \tag{7}$$

The pdf in Theorem 2 considers an Appell hypergeometric function F_4. We can write this as a series of hypergeometric functions ${}_2F_1$.

Figure 1 illustrates the pdf of Equation (7) for some parameters. When ρ increases, the largest values of y_1 and y_2 in the pdf are produced. However, these values depend on the other parameters. Independent of the ρ value, the pdf is close at the origin $(y_1, y_2) \approx (0, 0)$ with a positive bias, and it decays exponentially for the smallest values ($\nu = 4$, $\alpha = 4$, $q_1 = 4$, $q_2 = 4$, and $p = 4$). When $\rho = 0.9$ and $\nu, \alpha = 8$ or 12, for example, the pdf has more symmetry and variability but less bias. Note that the pdf of $\mathbf{Y}$, given in Equation (5), is symmetric for negative values of ρ. Specifically, the same representations hold for $\rho = -0.25, -0.5, -0.9$ from Figure 1 while keeping the other parameters fixed.

Theorem 3. *The joint cdf of* $\mathbf{Y} = (Y_1, Y_2)^\top$ *in Equation (7) can be expressed as*

$$\begin{aligned} F_{\mathbf{Y}}(Y_1 \le t_1, Y_2 \le t_2) &= \frac{\left(\frac{c^2t_1t_2}{q_1q_2}\right)^{p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\ &\times\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)^2_{k+m}\rho^{2k+2m}}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m(\nu/2+k)^2}\left(\frac{c^2t_1t_2}{q_1q_2}\right)^{pk} \\ &\times{}_2F_1\left(\frac{\nu+\alpha}{2}+k+m,\frac{\nu}{2}+k;\frac{\nu}{2}+k+1;-\left(\frac{ct_1}{q_1}\right)^p\right) \\ &\times{}_2F_1\left(\frac{\nu+\alpha}{2}+k+m,\frac{\nu}{2}+k;\frac{\nu}{2}+k+1;-\left(\frac{ct_2}{q_2}\right)^p\right). \end{aligned} \tag{8}$$

Proposition 1. *The joint mgf and characteristic functions of* $\mathbf{Y} = (Y_1, Y_2)^\top$ *given in Equation (7) are*

$$\begin{aligned} (a)\ M_{\mathbf{Y}}(t_1,t_2) &= \frac{\frac{(cp)^2}{q_1q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\ &\times\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}\left(\frac{\nu+\alpha}{2}\right)_{k+m}}{\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m k!m!}\rho^{4km}(c^2/q_1q_2)^{(p\nu/2+pk-1)} \\ &\times\left[\frac{1}{p}\left(\frac{q}{c}\right)^{p(\nu/2+k)}\frac{\Gamma(\nu/2+k)\Gamma(k+m+((\nu+\alpha)/2)}{\Gamma(m+((\nu+\alpha)/2)+\nu/2)}\right]^2 \\ &\left[-\frac{(t_1)^2-1}{(t_1)^4}\right]\left[-\frac{(t_2)^2-1}{(t_2)^4}\right], \end{aligned} \tag{9}$$

with $t < 0$ *and*

$$\begin{aligned} (b)\ \phi_{\mathbf{Y}}(t_1,t_2) &= \frac{\frac{(cp)^2}{q_1q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\ &\times\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}\left(\frac{\nu+\alpha}{2}\right)_{k+m}}{\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m k!m!}\rho^{4km}(c^2/q_1q_2)^{(p\nu/2+pk-1)} \end{aligned}$$

$$\times \left[\frac{1}{p} \left(\frac{q}{c} \right)^{p(\nu/2+k)} \frac{\Gamma(\nu/2+k)\Gamma(k+m+((\nu+\alpha)/2))}{\Gamma(m+((\nu+\alpha)/2)+\nu/2)} \right]^2$$
$$\times \left[-\frac{(it_1)^2-1}{(it_1)^4} \right] \left[-\frac{(it_2)^2-1}{(it_2)^4} \right]. \tag{10}$$

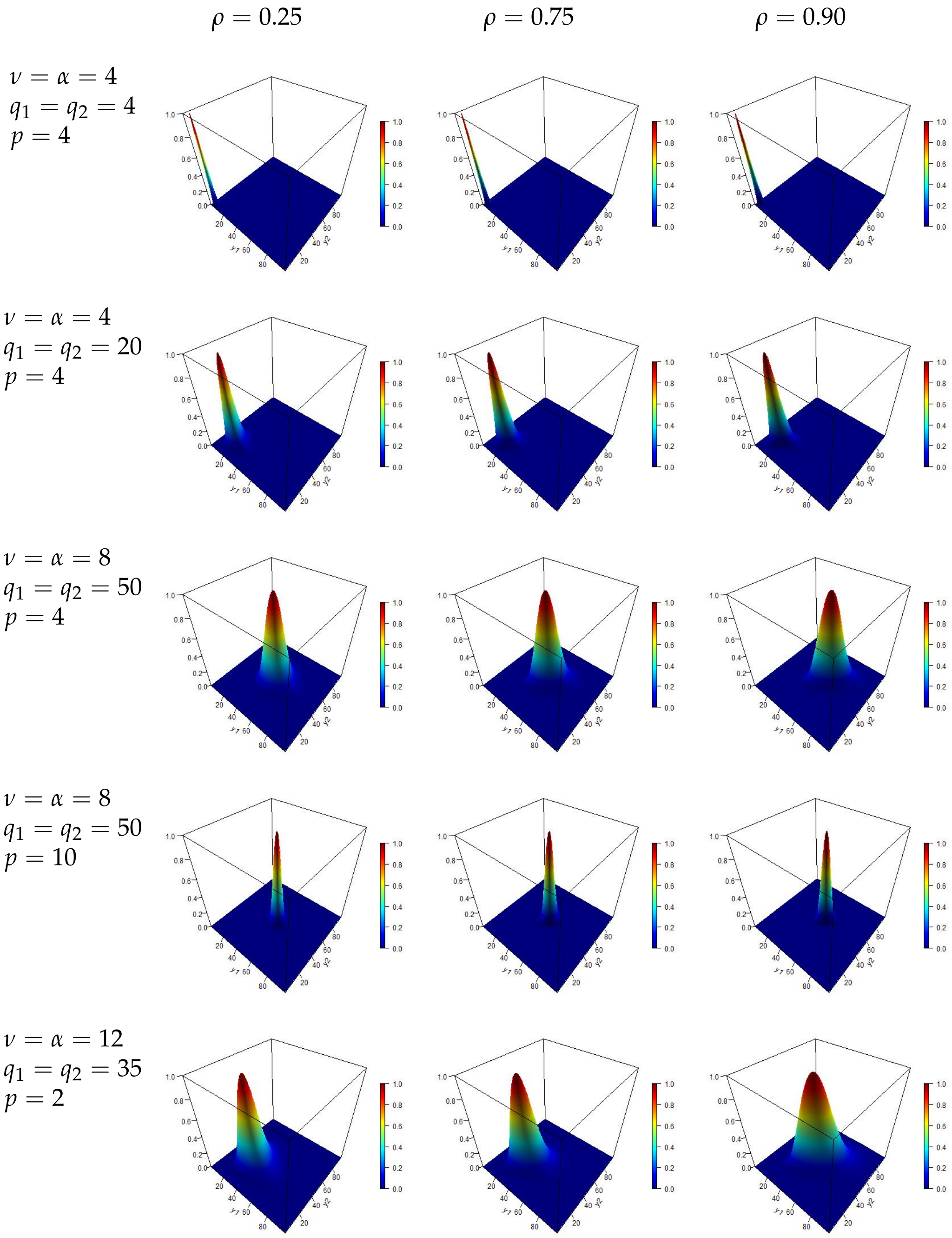

Figure 1. Bivariate pdf of Equation (7) for some parameter combinations.

Proposition 2. *The cross-product moment of* $\mathbf{Y} = (Y_1, Y_2)^\top$ *in Equation (7) can be expressed as*

$$\mathbb{E}[Y_1^a Y_2^b] = \frac{q_1^a q_2^b \Gamma\left(\frac{\alpha}{2} - \frac{a}{p}\right)\Gamma\left(\frac{\alpha}{2} - \frac{b}{p}\right)\Gamma\left(\frac{\nu}{2} + \frac{a}{p}\right)\Gamma\left(\frac{\nu}{2} + \frac{b}{p}\right)}{c^{a+b}\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)} \times {}_2F_1\left(-\frac{a}{p}, -\frac{b}{p}; \frac{\nu}{2}; \rho^2\right){}_2F_1\left(\frac{a}{p}, \frac{b}{p}; \frac{\alpha}{2}; \rho^2\right). \quad (11)$$

Proposition 2 illustrates that the cross-product moment is the product of two Gaussian hypergeometric functions. Corollary 1 is straightforward from Proposition 2, where the expected value and variance of a marginal Pareto–Feller random variable Y_k are presented as well as the covariance and correlation between two marginal Pareto–Feller random variables (Y_1 and Y_2).

Corollary 1. *If* $\mathbf{Y} = (Y_1, Y_2)^\top$*, then it has a pdf according to Equation (7). According to Proposition 2, we have the following:*

1. $\mathbb{E}[Y_i] = q_i, \ i = 1, 2.$
2. $Var(Y_i) = q_i^2 \left[\frac{\Gamma\left(\frac{\nu}{2}+\frac{2}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{2}{p}\right)}{c\Gamma\left(\frac{\nu}{2}+\frac{1}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{1}{p}\right)} - 1\right]$ *if* $\alpha p > 4$ *for* $i = 1, 2.$
3. $$Cov(Y_1, Y_2) = \frac{q^i 2\Gamma\left(\frac{\alpha}{2} - \frac{1}{p}\right)\Gamma\left(\frac{\alpha}{2} - \frac{1}{p}\right)\Gamma\left(\frac{\nu}{2} + \frac{1}{p}\right)\Gamma\left(\frac{\nu}{2} + \frac{1}{p}\right)}{c^2\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)} \times {}_2F_1\left(-\frac{1}{p}, -\frac{1}{p}; \frac{\nu}{2}; \rho^2\right){}_2F_1\left(\frac{1}{p}, \frac{1}{p}; \frac{\alpha}{2}; \rho^2\right). \quad (12)$$
4. *Let* $\rho_Y \equiv Corr(Y_1, Y_2)$*. Thus, we have*

$$\rho_Y = \frac{\Gamma^2\left(\frac{\nu}{2} + \frac{1}{p}\right)\Gamma^2\left(\frac{\alpha}{2} - \frac{1}{p}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\nu}{2} + \frac{2}{p}\right)\Gamma\left(\frac{\alpha}{2} - \frac{2}{p}\right) - \Gamma^2\left(\frac{\nu}{2} + \frac{1}{p}\right)\Gamma^2\left(\frac{\alpha}{2} - \frac{1}{p}\right)} \times \left[{}_2F_1\left(-\frac{1}{p}, -\frac{1}{p}; \frac{\nu}{2}; \rho^2\right){}_2F_1\left(\frac{1}{p}, \frac{1}{p}; \frac{\alpha}{2}; \rho^2\right) - 1\right]. \quad (13)$$

Figure 2 shows the correlation ρ_Y for some density parameters. When ν and α increase, ρ_Y increases. More specifically, when p increases, the correlation ρ_Y increases with small-to-large values of ν and α. When ρ increases, ρ_Y increases, as does its maximum value (from 0.05 to 0.80). Nevertheless, Corollary 1(4) illustrates that the correlation ρ_Y does not depend on either q or c.

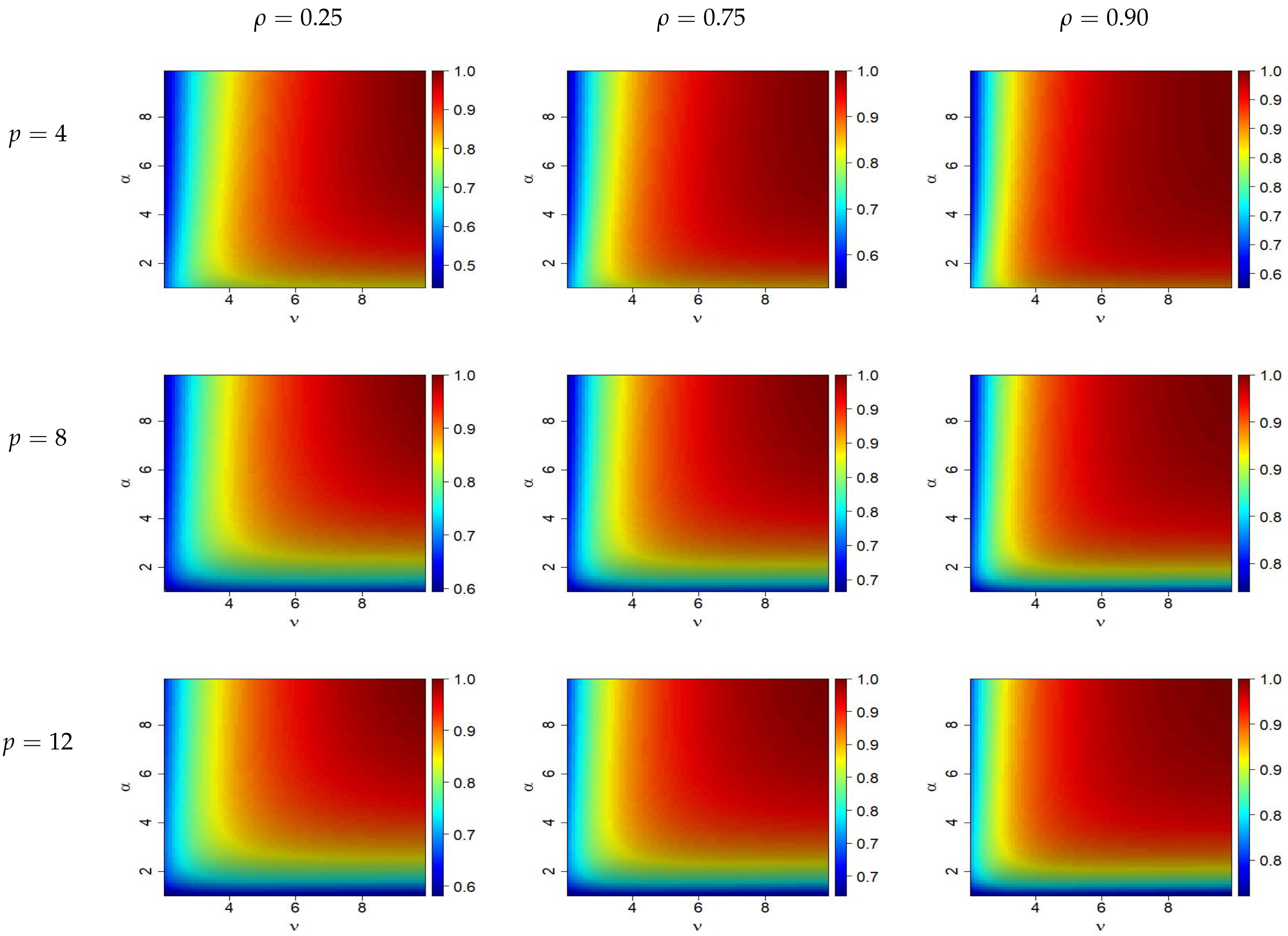

Figure 2. Correlation ρ_Y of Corollary 1(4) for some parameter combinations.

3. Differential Entropy and Mutual Information Index

The differential entropy of $\mathbf{Y}$ is an information uncertainty measure [27]. The differential entropy of $\mathbf{Y} = (Y_1, Y_2)^\top$ with a pdf $f_{\mathbf{Y}}(\mathbf{y})$ is

$$\begin{aligned} \mathcal{H}(\mathbf{Y}) &= -\mathbb{E}_{\mathbf{Y}}[\log\{f_{\mathbf{Y}}(\mathbf{Y})\}] \\ &= -\int_0^\infty \int_0^\infty f_{\mathbf{Y}}(\mathbf{y}) \log f_{\mathbf{Y}}(\mathbf{y}) dy_1 dy_2, \end{aligned} \tag{14}$$

and measures the contained information in $\mathbf{Y}$ based on its pdf's parameters.

On the other hand, the mutual information index (MII) [28] between Y_1 and Y_2 under a dependence assumption ($\rho \neq 0$) is

$$\begin{aligned} \mathcal{M}(\mathbf{Y}) &= \mathbb{E}\left[\log\left\{\frac{f_{\mathbf{Y}}(y_1, y_2)}{f_{Y_1}(y_1) f_{Y_2}(y_2)}\right\}\right] \\ &= \int_0^\infty \int_0^\infty f_{\mathbf{Y}}(y_1, y_2) \log\left\{\frac{f_{\mathbf{Y}}(y_1, y_2)}{f_{Y_1}(y_1) f_{Y_2}(y_2)}\right\} dy_1 dy_2. \end{aligned} \tag{15}$$

Proposition 3 ([29])**.** *Let $Y_i \sim PF(\mu, q_i/c, 1/p, \nu/2, \alpha/2)$. The entropy of Y_i ($i = 1, 2$) is*

$$\mathcal{H}(Y_i) = \log\left[\frac{cp}{q_i} \frac{\Gamma(\frac{\nu+\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\nu}{2})}\right] + \left(\frac{1}{p} - \frac{\alpha}{2}\right)\left[\psi\left(\frac{\alpha}{2}\right) - \psi\left(\frac{\nu}{2}\right)\right] + \left(\frac{\nu+\alpha}{2}\right)\left[\psi\left(\frac{\nu+\alpha}{2}\right) - \psi\left(\frac{\nu}{2}\right)\right],$$

where $\psi(X) = \frac{d \log \Gamma(x)}{dx}$ *is the digamma function.*

Proposition 4 ([30])**.** *Let* $x = u/n$*. We have that*

$$\log(1+x) = x + \mathcal{O}(n^{-2}),$$

as $n \to \infty$*.*

Proposition 5. *If* $\mathbf{Y} = (Y_1, Y_2)^\top$ *has the pdf given in Equation (7), then the following are true:*

(a) $\mathbb{E}_{\mathbf{Y}}[Y_i^p] = \frac{q_i^p \Gamma(\frac{\nu}{2}+1)\Gamma(\frac{\alpha}{2}-1)}{c^p \Gamma(\frac{\nu}{2})\Gamma(\frac{\alpha}{2})}$.

(b) $\mathbb{E}_{\mathbf{Y}}[\log(Y_i)] = \frac{1}{p(1-\rho^2)^{-(\nu+\alpha)/2}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\frac{\nu}{2})_k (\frac{\alpha}{2})_m \rho^{2k+2m}}{k!m!} \left\{ \psi(\frac{\nu}{2}+k) - \psi(\frac{\alpha}{2}+m) - \log\left(\frac{c}{q_i}\right)^p \right\}$.

Proposition 6. *An approximation of the differential entropy of* $\mathbf{Y} = (Y_1, Y_2)^\top$ *is*

$$\begin{aligned}
\mathcal{H}(\mathbf{Y}) \approx & -\log\left[\frac{\frac{(cp)^2}{q_1 q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\right] \\
& + \left(1 - \frac{p\nu}{2}\right)\left\{\log\left(\frac{c^2}{q_1 q_2}\right) + \mathbb{E}_{\mathbf{Y}}[\log Y_1] + \mathbb{E}_{\mathbf{Y}}[\log Y_2]\right\} \\
& + c^p\left(\frac{\nu+\alpha}{2}\right)\left\{\frac{\mathbb{E}_{\mathbf{Y}}[Y_1^p]}{q_1^p} + \frac{\mathbb{E}_{\mathbf{Y}}[Y_2^p]}{q_2^p}\right\},
\end{aligned}$$

where $\mathbb{E}_{\mathbf{Y}}[Y_i^p]$ *and* $\mathbb{E}_{\mathbf{Y}}[\log Y_i]$, $i = 1, 2$ *are obtained from parts (a) and (b) of Proposition 5, respectively.*

From Equation (15), an MII between Y_1 and Y_2 is expressed in terms of the marginal and joint differential entropies [28,31]. Then, using Propositions 3 and 6, the MII between Y_1 and Y_2 can be approximated as follows:

$$\begin{aligned}
\mathcal{M}(\mathbf{Y}) = & \; \mathcal{H}(Y_1) + \mathcal{H}(Y_2) - \mathcal{H}(\mathbf{Y}) \\
\approx & \; 2\log\left[cp\frac{\Gamma(\frac{\nu+\alpha}{2})\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\nu}{2})}\right] - \log(q_1 q_2) + \log\left[\frac{\frac{(cp)^2}{q_1 q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\right] \\
& -\left(1 - \frac{p\nu}{2}\right)\left\{\log\left(\frac{c^2}{q_1 q_2}\right) + \mathbb{E}_{\mathbf{Y}}[\log Y_1] + \mathbb{E}_{\mathbf{Y}}[\log Y_2]\right\} \\
& -c^p\left(\frac{\nu+\alpha}{2}\right)\left\{\frac{\mathbb{E}[Y_1^p]}{q_1^p} + \frac{\mathbb{E}[Y_2^p]}{q_2^p}\right\} + \left(\frac{2}{p} - \alpha\right)\left[\psi\left(\frac{\alpha}{2}\right) - \psi\left(\frac{\nu}{2}\right)\right] \\
& +(\nu+\alpha)\left[\psi\left(\frac{\nu+\alpha}{2}\right) - \psi\left(\frac{\nu}{2}\right)\right].
\end{aligned}$$

One particular case is when $p = 1$. Thus, using Corollary 1(1), we have

$$\begin{aligned}
\mathcal{M}(\mathbf{Y}) \approx & \left(1 + \frac{\nu}{2}\right)\log\left(\frac{c^2}{q_1 q_2}\right) + 2\log\left[\frac{\Gamma^2(\frac{\nu+\alpha}{2})}{\Gamma^2(\frac{\nu}{2})}\right] \\
& -\left(\frac{\nu+\alpha}{2}\right)\log(1-\rho^2) + (2-\alpha)\left[\psi\left(\frac{\alpha}{2}\right) - \psi\left(\frac{\nu}{2}\right)\right] \\
& +(\nu+\alpha)\left[\psi\left(\frac{\nu+\alpha}{2}\right) - \psi\left(\frac{\nu}{2}\right) - c\right] \\
& -\left(1 - \frac{\nu}{2}\right)\{\mathbb{E}_{\mathbf{Y}}[\log Y_1] + \mathbb{E}_{\mathbf{Y}}[\log Y_2]\}. \qquad (16)
\end{aligned}$$

Figure 3 illustrates the behavior of the approximated MII obtained in Equation (16) while assuming several values for $\mathbf{Y}$ and $p = 1$. We observed that $\mathcal{M}(\mathbf{Y}) > 0$ and increased for $q_i \to 0$. As in the correlation function case (Figure 2), the MII increased when the correlation parameter ρ increased.

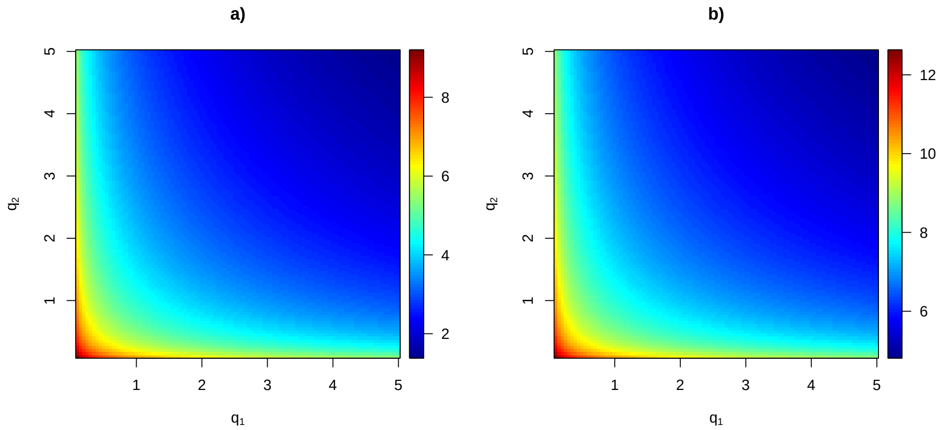

Figure 3. Approximated mutual information index of Pareto–Feller distribution assuming $p = 1$, $\alpha = 5$, $\nu = 4$, (**a**) $\rho = 0.25$, and (**b**) $\rho = 0.75$.

4. Concluding Remarks

We presented a representation of the Pareto–Feller distribution with a scale mixture of two gamma random variables. The respective stochastic representation was obtained by the quotient of a scale mixture of two gamma random variables. Then, the resulting bivariate density considered the products of two confluent hypergeometric functions. In particular, the probability distribution function, cumulative distribution function, moment generation function, covariance function, correlation function, cross-product moments, and approximations for the differential entropy and, as a consequence, the mutual information index were derived. Some numerical examples illustrated the behavior of the provided expressions.

Some inferential aspects can be addressed in a future work, such as (1) a numerical approach for optimization of the log-likelihood function; (2) the pseudo-likelihood method, considering the optimization of an objective function which depends on a bivariate pdf; (3) the model's identifiability; (4) a Bayesian approach; and (5) an extention to the multivariate case. We also encourage researching the consideration of a Pareto–Feller distribution in modeling nonnegative bivariate data.

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Appendix A

Proof of Theorem 1. Following [10], note that the bivariate distribution of the gamma random vector can be rewritten in terms of the hypergeometric function using the identity ${}_0F_1(;b;x) = \Gamma(b)x^{(1-b)/2}I_{b-1}(2\sqrt{x})$, given by

$$f_{\mathbf{H}}(\mathbf{h}) = \frac{(h_1h_2)^{\delta/2-1}e^{-\frac{(h_1+h_2)}{(1-\rho^2)}}}{\Gamma^2\left(\frac{\delta}{2}\right)(1-\rho^2)^{\delta/2}}\,{}_0F_1\left(\frac{\delta}{2};\frac{\rho^2h_1h_2}{(1-\rho^2)^2}\right) \tag{A1}$$

Under the transformations $w_1 = v_1r_1$ and $w_2 = v_2r_2$ in Equation (A1) with the Jacobian $J((w_1,w_2)\to(v_1,v_2)) = r_1r_2$, and by using series expansion of the hypergeometric function ${}_0F_1$, we have

$$\begin{aligned} f_{\mathbf{V}}(\mathbf{v}) &= \int_{\mathbb{R}_+^2} f_{\mathbf{W}|\mathbf{R}}(\mathbf{w}|\mathbf{r})f_{\mathbf{R}}(\mathbf{r})Jd\mathbf{r} \\ &= \int_{\mathbb{R}_+^2} \frac{(v_1v_2)^{\nu/2-1}(r_1r_2)^{(\nu+\alpha)/2-1}e^{-\frac{(v_1r_1+v_2r_2)}{(1-\rho^2)}}e^{-\frac{(r_1+r_2)}{(1-\rho^2)}}}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{(\nu+\alpha)/2}} \\ &\quad \times {}_0F_1\left(\frac{\nu}{2};\frac{\rho^2v_1v_2r_1r_2}{(1-\rho^2)^2}\right){}_0F_1\left(\frac{\alpha}{2};\frac{\rho^2r_1r_2}{(1-\rho^2)^2}\right)d\mathbf{r} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_{\mathbb{R}_+^2} \frac{(v_1v_2)^{\nu/2-1}(r_1r_2)^{(\nu+\alpha)/2-1}e^{-\frac{(v_1+1)r_1}{(1-\rho^2)}}e^{-\frac{(v_2+1)r_2}{(1-\rho^2)}}}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{(\nu+\alpha)/2}k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m} \\ &\quad \times \left(\frac{\rho^2(\mathbf{m}h)v_1v_2r_1r_2}{(1-\rho^2)^2}\right)^k\left(\frac{\rho^2r_1r_2}{(1-\rho^2)^2}\right)^m d\mathbf{r} \\ &= \frac{(v_1v_2)^{\nu/2-1}}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{(\nu+\alpha)/2}} \\ &\quad \times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{I(k,m)}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\left(\frac{\rho^2v_1v_2}{(1-\rho^2)^2}\right)^k\left(\frac{\rho^2}{(1-\rho^2)^2}\right)^m, \end{aligned} \tag{A2}$$

where, using Fubini's theorem and Equation (3.381.4) in [32], we obtain

$$\begin{aligned} I(k,m) &= \int_{\mathbb{R}_+} r_1^{(\nu+\alpha)/2+k+m-1}e^{-\frac{(v_1+1)r_1}{(1-\rho^2)}}dr_i\int_{\mathbb{R}_+} r_2^{(\nu+\alpha)/2+k+m-1}e^{-\frac{(v_2+1)r_2}{(1-\rho^2)}}dr_2 \\ &= \Gamma^2\left(\frac{\nu+\alpha}{2}+k+m\right)\left[\frac{(1-\rho^2)}{v_1+1}\right]^{\frac{\nu+\alpha}{2}+k+m}\left[\frac{(1-\rho^2)}{v_2+1}\right]^{\frac{\nu+\alpha}{2}+k+m} \end{aligned} \tag{A3}$$

In addition, by combining Equations (A3) and (A2), we obtain

$$\begin{aligned} f_{\mathbf{V}_{12}}(\mathbf{v}_{12}) &= \frac{(v_1v_2)^{\nu/2-1}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)[(v_1+1)(v_2+1)]^{-(\nu+\alpha)/2}}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\ &\quad \times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^2}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\left(\frac{\rho^2v_1v_2}{(v_1+1)(v_2+1)}\right)^k\left(\frac{\rho^2}{(v_1+1)(v_2+1)}\right)^m \\ &= \frac{(v_1v_2)^{\nu/2-1}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)[(v_1+1)(v_2+1)]^{-(\nu+\alpha)/2}}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\ &\quad \times F_4\left(\frac{\nu+\alpha}{2},\frac{\nu+\alpha}{2};\frac{\nu}{2},\frac{\alpha}{2};\frac{\rho^2v_1v_2}{(v_1+1)(v_2+1)},\frac{\rho^2}{(v_1+1)(v_2+1)}\right) \end{aligned}$$

□

Proof of Theorem 2. Under the transformations $v_1 = (y_1/q_1)^p$ and $v_2 = (y_2/q_2)^p$ in Equation (5) with a Jacobian $J((v_1;v_2) \to (y_1;y_2)) = (p/q)^2(y_1y_2/(q_1q_2))^{p-1}$, the pdf of $\mathbf{Y}$ is given by Equation (7). □

Proof of Theorem 3. Using series expansion of the Appell hypergeometric function, we obtain

$$\begin{aligned}
&F_{\mathbf{Y}}(Y_1 \le t_1, Y_2 \le t_2) \\
= &\frac{p^2c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \int_0^{t_1}\int_0^{t_2} (y_1y_2)^{p\nu/2-1}\left[\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)\right]^{-\frac{(\nu+\alpha)}{2}} \\
\times &F_4\left(\frac{\nu+\alpha}{2},\frac{\nu+\alpha}{2};\frac{\nu}{2},\frac{\alpha}{2};\frac{\rho^2(c^2y_1y_2)^p(q_1q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)},\frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)d\mathbf{y} \\
= &\frac{p^2c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \int_0^{t_1}\int_0^{t_2} (y_1y_2)^{p\nu/2-1}\left[\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)\right]^{-\frac{(\nu+\alpha)}{2}} \\
&\times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)^2_{k+m}}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m} \\
&\times \left(\frac{\rho^2(c^2y_1y_2)^p(q_1q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)^k\left(\frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)^m d\mathbf{y} \\
= &\frac{p^2c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)^2_{k+m}\rho^{2m}I(k,m)}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\left(\frac{c^{2p}\rho^2}{(q_1q_2)^p}\right)^k
\end{aligned} \tag{A4}$$

Let $z_1 = y_1^p$ and $z_2 = y_2^p$ be transformations in Equation (A4), where $0 < z_i < t_i^p$, $i = 1,2$. Using Fubini's theorem and Equation (3.1941.1) in [32], we obtain

$$\begin{aligned}
I(k,m) &= \frac{1}{p^2}\int_0^{t_1^p} z_1^{\nu/2+k-1}\left[\left(\frac{c}{q_1}\right)^p z_1+1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)}dz_1 \\
&\quad\times\int_0^{t_2^p} z_2^{\nu/2+k-1}\left[\left(\frac{c}{q_2}\right)^p z_2+1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)}dz_2 \\
&= \frac{(t_1t_2)^{p\nu/2+pk}}{p^2(\nu/2+k)^2}\,{}_2F_1\left(\frac{\nu+\alpha}{2}+k+m,\frac{\nu}{2}+k;\frac{\nu}{2}+k+1;-\left(\frac{ct_1}{q_1}\right)^p\right) \\
&\quad {}_2F_1\left(\frac{\nu+\alpha}{2}+k+m,\frac{\nu}{2}+k;\frac{\nu}{2}+k+1;-\left(\frac{ct_2}{q_2}\right)^p\right)
\end{aligned} \tag{A5}$$

In addition, by combining Equations (A4) and (A5), we obtain Equation (8). □

Proof of Proposition 1. For part (a), by using the definition of the mgf and series expansion of the Appell hypergeometric function, we obtain

$$\begin{aligned}
&M_{\mathbf{Y}}(t_1,t_2) \\
= &\frac{p^2c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\
&\times\int_{\mathbb{R}^2_+} e^{t_1y_1+t_2y_2}y_1^{p\nu/2-1}y_2^{p\nu/2-1}\left[\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)\right]^{-\frac{(\nu+\alpha)}{2}}
\end{aligned}$$

$$\times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^{2}}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}$$

$$\times \left(\frac{\rho^2 (c^2 y_1 y_2)^p (q_1 q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p + 1\right)\left(\left(\frac{cy_2}{q_2}\right)^p + 1\right)} \right)^k \left(\frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p + 1\right)\left(\left(\frac{cy_2}{q_2}\right)^p + 1\right)} \right)^m d\mathbf{y}$$

$$= \frac{p^2 c^{p\nu} (q_1 q_2)^{-p\nu/2} \Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^{2} \rho^{2m} I(k,m)}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m} \left(\frac{c^{2p}\rho^2}{(q_1 q_2)^p}\right)^k. \quad \text{(A6)}$$

Using Fubini's theorem in Equation (A6), we obtain

$$\begin{aligned}
I(k,m) &= \int_0^\infty e^{t_1 y_1} y_1^{(p\nu/2)+pk-1} \left[\left(\frac{cy_1}{q_1}\right)^p + 1\right]^{-\left(\frac{(\nu+\alpha)}{2}+k+m\right)} dy_1 \\
&\times \int_0^\infty e^{t_2 y_2} y_2^{(p\nu/2)+pk-1} \left[\left(\frac{cy_2}{q_2}\right)^p + 1\right]^{-\left(\frac{(\nu+\alpha)}{2}+k+m\right)} dy_2 \\
&= \left[\frac{1}{p}\left(\frac{q}{c}\right)^{p(\nu/2+k)} \frac{\Gamma(\nu/2+k)\Gamma(k+m+((\nu+\alpha)/2))}{\Gamma(m+((\nu+\alpha)/2)+\nu/2)}\right] \left[e^{(t_1 y_1)} - \frac{1}{(t_1)^2}\right]\Bigg|_{y_1=0}^{y_1=\infty} \\
&\times \left[\frac{1}{p}\left(\frac{q}{c}\right)^{p(\nu/2+k)} \frac{\Gamma(\nu/2+k)\Gamma(k+m+((\nu+\alpha)/2))}{\Gamma(m+((\nu+\alpha)/2)+\nu/2)}\right] \left[e^{(t_2 y_2)} - \frac{1}{(t_2)^2}\right]\Bigg|_{y_2=0}^{y_2=\infty} \\
&= \left[\frac{1}{p}\left(\frac{q}{c}\right)^{p(\nu/2+k)} \frac{\Gamma(\nu/2+k)\Gamma(k+m+((\nu+\alpha)/2))}{\Gamma(m+((\nu+\alpha)/2)+\nu/2)}\right]^2 \left[-\frac{1}{(t_i)^2}\right]\left[1 - \frac{1}{(t_i)^2}\right].
\end{aligned} \quad \text{(A7)}$$

Note that if $f(y) = e^{t_i y_i}$ and

$$g(y) = (y_i)^{(p\nu/2)+pk-1} \left[\left(\frac{c}{q_i}\right)^p (y_i)^p + 1\right]^{-(k+m+(\nu+\alpha)/2)} dy_i,$$

then we can apply integration by substitution for $\int_0^\infty f(y)g(y)dy$. With the change in variables $u_i = e^{t_i y_i} \Rightarrow du_i = t_i e^{t_i y_i}$, and by letting

$$dv_i = (y_i)^{(p\nu/2)+pk-1} \left[\left(\frac{c}{q_i}\right)^p (y_i)^p + 1\right]^{-(k+m+(\nu+\alpha)/2)} dy_i,$$

then $v_i = \int_0^\infty dv_i$, where Equation (3.241.4.11) in [32] was used with $t_i < 0$, $i = 1,2$. By combining Equations (A6) and (A7), we obtain

$$\begin{aligned}
M_{\mathbf{Y}}(t_1,t_2) &= \frac{\frac{(cp)^2}{q_1 q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \\
&\times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}\left(\frac{\nu+\alpha}{2}\right)_{k+m}}{\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m k!m!} \rho^{4km} (c^2/q_1 q_2)^{(p\nu/2+pk-1)} \\
&\times \left[\frac{1}{p}\left(\frac{q}{c}\right)^{p(\nu/2+k)} \frac{\Gamma(\nu/2+k)\Gamma(k+m+((\nu+\alpha)/2))}{\Gamma(m+((\nu+\alpha)/2)+\nu/2)}\right]^2 \\
&\times \left[-\frac{(t_1)^2-1}{(t_1)^4}\right]\left[-\frac{(t_2)^2-1}{(t_2)^4}\right].
\end{aligned}$$

For part (b), and by following the proof of part (a), the proof of the characteristic function of **Y** is straightforward. □

Proof of Proposition 2. By the definition of cross-product moment, and using series expansion of the Appell hypergeometric function of the fourth kind, we obtain

$$
\begin{aligned}
&\mathbb{E}(Y_1^a Y_2^b)\\
=\;& \frac{p^2 c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\\
&\int_{\mathbb{R}_+^2} y_1^{p\nu/2+a-1} y_2^{p\nu/2+b-1}\left[\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)\right]^{-\frac{(\nu+\alpha)}{2}}\\
\times\;& F_4\left(\frac{\nu+\alpha}{2},\frac{\nu+\alpha}{2};\frac{\nu}{2},\frac{\alpha}{2};\frac{\rho^2(c^2y_1y_2)^p(q_1q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)},\frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)d\mathbf{y}\\
=\;& \frac{p^2 c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\\
&\times\int_{\mathbb{R}_+^2} y_1^{p\nu/2+a-1} y_2^{p\nu/2+b-1}\left[\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)\right]^{-\frac{(\nu+\alpha)}{2}}\\
\times\;& \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^2}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\\
&\times\left(\frac{\rho^2(c^2y_1y_2)^p(q_1q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)^k\left(\frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)^m d\mathbf{y}\\
=\;& \frac{p^2 c^{p\nu}(q_1q_2)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^2\rho^{2m}I(k,m)}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\left(\frac{c^{2p}\rho^2}{(q_1q_2)^p}\right)^k, \qquad \text{(A8)}
\end{aligned}
$$

Using Fubini's theorem and Equation (3.241.411) from [32] in Equation (A8), we obtain

$$
\begin{aligned}
I(k,m) &= \int_{\mathbb{R}_+} y_1^{p\nu/2+a+pk-1}\left[\left(\frac{cy_1}{q_1}\right)^p+1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)}dy_1\\
&\quad\times\int_{\mathbb{R}_+} y_2^{p\nu/2+b+pk-1}\left[\left(\frac{cy_2}{q_2}\right)^p+1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)}dy_2\\
&= \frac{\Gamma\left(\frac{\nu}{2}+\frac{a}{p}+k\right)\Gamma\left(\frac{\alpha}{2}-\frac{a}{p}+m\right)\Gamma\left(\frac{\nu}{2}+\frac{b}{p}+k\right)\Gamma\left(\frac{\alpha}{2}-\frac{b}{p}+m\right)}{p^2\Gamma^2\left(\frac{\nu+\alpha}{2}+k+m\right)}\\
&\quad\times\left(\frac{q_1}{c}\right)^{\frac{p\nu}{2}+a+pk}\left(\frac{q_2}{c}\right)^{\frac{p\nu}{2}+b+pk}. \qquad \text{(A9)}
\end{aligned}
$$

By combining Equations (A8) and (A9), we obtain

$$
\begin{aligned}
\mathbb{E}(Y_1^aY_2^b) &= \frac{q_1^aq_2^b\Gamma\left(\frac{\nu}{2}+\frac{a}{p}\right)\Gamma\left(\frac{\nu}{2}+\frac{b}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{a}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{b}{p}\right)}{c^{a+b}\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-\frac{\nu+\alpha}{2}}}\\
&\quad\times\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu}{2}+\frac{a}{p}\right)_k\left(\frac{\nu}{2}+\frac{b}{p}\right)_k\left(\frac{\alpha}{2}-\frac{a}{p}\right)_m\left(\frac{\alpha}{2}-\frac{b}{p}\right)_m}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_k}\rho^{2(k+m)}\\
&= \frac{q_1^aq_2^b\Gamma\left(\frac{\nu}{2}+\frac{a}{p}\right)\Gamma\left(\frac{\nu}{2}+\frac{b}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{a}{p}\right)\Gamma\left(\frac{\alpha}{2}-\frac{b}{p}\right)}{c^{a+b}\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-\frac{\nu+\alpha}{2}}}\\
&\quad\times{}_2F_1\left(\frac{\nu}{2}+\frac{a}{p},\frac{\nu}{2}+\frac{b}{p};\frac{\nu}{2};\rho^2\right){}_2F_1\left(\frac{\alpha}{2}-\frac{a}{p},\frac{\alpha}{2}-\frac{b}{p};\frac{\alpha}{2};\rho^2\right)
\end{aligned}
$$

Finally, using a Euler transformation, we have Equation (11). □

Proof of Proposition 5. For part (a), we have

$$\mathbb{E}_{\mathbf{Y}}(Y_i^p) = \frac{p^2 c^{p\nu}(q_i q_j)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^2 \rho^{2m} I(k,m)}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\left(\frac{c^{2p}\rho^2}{(q_i q_j)^p}\right)^k. \quad \text{(A10)}$$

By using Fubini's theorem and Equation (3.241.411) from [32] in Equation (A10), we obtain

$$\begin{aligned} I(k,m) &= \int_{\mathbb{R}_+} y_i^{p\nu/2+pk+p-1}\left[\left(\frac{c y_i}{q_i}\right)^p + 1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)} dy_i \\ &\times \int_{\mathbb{R}_+} y_j^{p\nu/2+pk-1}\left[\left(\frac{c y_j}{q_j}\right)^p + 1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)} dy_j \\ &= \frac{\Gamma\left(\frac{\nu}{2}+k+1\right)\Gamma\left(\frac{\alpha}{2}+m-1\right)\Gamma\left(\frac{\nu}{2}+k\right)\Gamma\left(\frac{\alpha}{2}+m\right)}{p^2\Gamma^2\left(\frac{\nu+\alpha}{2}+k+m\right)}\left(\frac{q_i}{c}\right)^{\frac{p\nu}{2}+pk+p}\left(\frac{q_j}{c}\right)^{\frac{p\nu}{2}+pk}. \end{aligned} \quad \text{(A11)}$$

By combining Equations (A10) and (A11), we obtain

$$\begin{aligned} \mathbb{E}_{\mathbf{Y}}(Y_i^p) &= \frac{q_i^p\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(\frac{\alpha}{2}-1\right)}{c^p\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-\frac{\nu+\alpha}{2}}} \\ &\times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu}{2}+1\right)_k\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}-1\right)_m\left(\frac{\alpha}{2}\right)_m}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_k}\rho^{2(k+m)} \\ &= \frac{q_i^p\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(\frac{\alpha}{2}-1\right)}{c^p\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-\frac{\nu+\alpha}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+1,\frac{\nu}{2};\frac{\nu}{2};\rho^2\right){}_2F_1\left(\frac{\alpha}{2}-1,\frac{\alpha}{2};\frac{\alpha}{2};\rho^2\right). \end{aligned}$$

Using the identity ${}_2F_1(a,b;b;x) = (1-x)^{-a}$ in the last equality, we obtain

$$\mathbb{E}_{\mathbf{Y}}(Y_i^p) = \frac{q_i^p\Gamma\left(\frac{\nu}{2}+1\right)\Gamma\left(\frac{\alpha}{2}-1\right)}{c^p\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}.$$

For part (b), we have

$$\mathbb{E}_{\mathbf{Y}}(\log Y_i) = \frac{p^2 c^{p\nu}(q_i q_j)^{-p\nu/2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}} \times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{\nu+\alpha}{2}\right)_{k+m}^2 \rho^{2m} I(k,m)}{k!m!\left(\frac{\nu}{2}\right)_k\left(\frac{\alpha}{2}\right)_m}\left(\frac{c^{2p}\rho^2}{(q_i q_j)^p}\right)^k. \quad \text{(A12)}$$

Using Fubini's theorem and Equation (3.241.411) from [32] in Equation (A12), we obtain

$$\begin{aligned} I(k,m) &= \int_{\mathbb{R}_+} \log(y_i) y_i^{p\nu/2+pk-1}\left[\left(\frac{c y_i}{q_i}\right)^p + 1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)} dy_i \\ &\times \int_{\mathbb{R}_+} y_j^{p\nu/2+pk-1}\left[\left(\frac{c y_j}{q_j}\right)^p + 1\right]^{-\left(\frac{\nu+\alpha}{2}+k+m\right)} dy_j \\ &= \frac{\Gamma^2\left(\frac{\nu}{2}+k\right)\Gamma^2\left(\frac{\alpha}{2}+m\right)}{p^3\Gamma^2\left(\frac{\nu+\alpha}{2}+k+m\right)}\left(\frac{q_i}{c}\right)^{\frac{p\nu}{2}+pk}\left(\frac{q_j}{c}\right)^{\frac{p\nu}{2}+pk} \end{aligned}$$

$$\times \left\{ \psi\left(\frac{\nu}{2}+k\right) - \psi\left(\frac{\alpha}{2}+m\right) - \log\left(\frac{c}{q_i}\right)^p \right\}. \quad \text{(A13)}$$

By combining Equations (A12) and (A13), we obtain

$$\begin{aligned} \mathbb{E}_{\mathbf{Y}}(\log Y_i) &= \frac{1}{p(1-\rho^2)^{-(\nu+\alpha)/2}} \\ &\quad \times \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\left(\frac{\nu}{2}\right)_k \left(\frac{\alpha}{2}\right)_m \rho^{2k+2m}}{k!m!} \left\{ \psi\left(\frac{\nu}{2}+k\right) - \psi\left(\frac{\alpha}{2}+m\right) - \log\left(\frac{c}{q_i}\right)^p \right\}. \end{aligned}$$

□

Proof of Proposition 6. In evaluating the density of Theorem 2 in Equation (14), we have

$$\begin{aligned} &\mathcal{H}(\mathbf{Y}) \\ &= -\log\left[\frac{\frac{(cp)^2}{q_1q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\right] - \left(\frac{p\nu}{2}-1\right)\left[\log\left(\frac{c^2}{q_1q_2}\right) + \mathbb{E}_{\mathbf{Y}}[\log Y_1] + \mathbb{E}_{\mathbf{Y}}[\log Y_2]\right] \\ &+ \left(\frac{\nu+\alpha}{2}\right)\left\{\mathbb{E}_{\mathbf{Y}}\left[\log\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\right] + \mathbb{E}_{\mathbf{Y}}\left[\log\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)\right]\right\} \\ &- \mathbb{E}_{\mathbf{Y}}\left[\log F_4\left(\frac{\nu+\alpha}{2}, \frac{\nu+\alpha}{2}; \frac{\nu}{2}, \frac{\alpha}{2}; \frac{\rho^2(c^2y_1y_2)^p(q_1q_2)^{-p}}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}, \frac{\rho^2}{\left(\left(\frac{cy_1}{q_1}\right)^p+1\right)\left(\left(\frac{cy_2}{q_2}\right)^p+1\right)}\right)\right]. \end{aligned}$$

By assuming in the last Appell hypergeometric function that its sum converges at the first term (i.e., when $k = m = 0$) [14], we have that $\mathbb{E}_{\mathbf{Y}}[\log F_4(\cdot)] \approx 0$. Aside from this, considering Proposition 4, we have that

$$\mathbb{E}_{\mathbf{Y}}\left[\log\left(\left(\frac{cY_i}{q_i}\right)^p+1\right)\right] \approx \left(\frac{c}{q_i}\right)^p \mathbb{E}_{\mathbf{Y}}[Y_i^p], \quad i = 1, 2.$$

Then, the differential entropy of $\mathbf{Y}$ can be approximated by

$$\begin{aligned} \mathcal{H}(\mathbf{Y}) \approx &-\log\left[\frac{\frac{(cp)^2}{q_1q_2}\Gamma^2\left(\frac{\nu+\alpha}{2}\right)}{\Gamma^2\left(\frac{\nu}{2}\right)\Gamma^2\left(\frac{\alpha}{2}\right)(1-\rho^2)^{-(\nu+\alpha)/2}}\right] \\ &- \left(\frac{p\nu}{2}-1\right)\left[\log\left(\frac{c^2}{q_1q_2}\right) + \mathbb{E}_{\mathbf{Y}}[\log Y_1] + \mathbb{E}_{\mathbf{Y}}[\log Y_2]\right] \\ &+ c^p\left(\frac{\nu+\alpha}{2}\right)\left\{\frac{\mathbb{E}_{\mathbf{Y}}[Y_1^p]}{q_1^p} + \frac{\mathbb{E}_{\mathbf{Y}}[Y_2^p]}{q_2^p}\right\}. \end{aligned}$$

This concludes the proof. □

References

1. Johnson, N.; Kotz, S.; Balakrishnan, N. *Continuous Univariate Distributions*, 2nd ed.; John Wiley & Sons Ltd.: New York, NY, USA, 1994; Volume 1.
2. Arnold, B. *Pareto Distributions*, 2nd ed.; Chapman and Hall/CRC: Boca Raton, FL, USA, 2015.
3. Feller, W. *An Introduction to Probability Theory and Its Applications*, 2nd ed.; John Wiley & Sons: Hoboken, NJ, USA, 1971; Volume 2.
4. Pickands, J. Statistical Inference Using Extreme Order Statistics. *Ann. Stat.* **1975**, *3*, 119–131.
5. Lomax, K.S. Business failures: Another example of the analysis of failure data. *J. Am. Stat. Assoc.* **1954**, *49*, 847–852. [CrossRef]
6. Arnold, B. *Pareto Distributions*; International Cooperative Publishing House: Fairland, MD, USA, 1983.
7. Cohen, C.A. *Truncated and Censored Samples: Theory and Applications*, 1st ed.; CRC Press: Boca Raton, FL, USA, 1991.
8. Gupta, R.C.; Akman, O.; Lvin, S. A study of log—Logistic model in survival analysis. *Biometrical J.* **1999**, *41*, 431–443. [CrossRef]
9. Bevilacqua, M.; Caamaño-Carrillo, C.; Arellano-Valle, R.B.; Morales-Oñate, V. Non-Gaussian geostatistical modeling using (skew) T processes. *Scand. J. Stat.* **2021**, *48*, 212–245. [CrossRef]

10. Bevilacqua, M.; Caamaño-Carrillo, C.; Gaetan, C. On modeling positive continuous data with spatiotemporal dependence. *Environmetrics* **2020**, *31*, e2632. [CrossRef]
11. Morales-Navarrete, D.; Bevilacqua, M.; Caamaño-Carrillo, C.; Castro, L.M. Modeling point referenced spatial count data: A Poisson process approach. *J. Am. Stat. Assoc.* **2022**, *119*, 1–14. [CrossRef]
12. Kotz, S.; Balakrishnan, N.; Johnson, N.L. *Continuous Multivariate Distributions, Volume 1: Models and Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2004.
13. Vere-Jones, D. Ergodic properties of nonnegative matrices. I. *Pac. J. Math.* **1967**, *22*, 361–386. [CrossRef]
14. Caamaño-Carrillo, C.; Contreras-Reyes, J.E. A generalization of the bivariate gamma distribution based on generalized hypergeometric functions. *Mathematics* **2022**, *10*, 1502. [CrossRef]
15. Caamano-Carrillo, C.; Contreras-Reyes, J.E.; González-Navarrete, M.; Sánchez, E. Bivariate superstatistics based on generalized gamma distribution. *Eur. Phys. J. B* **2020**, *93*, 43. [CrossRef]
16. Krishnamoorthy, A.S.; Parthasarthy, M.D. A multivariate gamma-type distribution. *Ann. Math. Stat.* **1951**, *22*, 549–557. [CrossRef]
17. Krishnaiah, P.R.; Rao, M.M. Remarks on a multivariate gamma distribution. *Math. Assoc. Am.* **1961**, *68*, 342–346. [CrossRef]
18. Royen, T. Integral representations and approximations for multivariate gamma distributions. *Ann. Inst. Stat. Math.* **2004**, *59*, 499–513. [CrossRef]
19. Kotz, S.; Nadarajah, S. *Extreme Value Distributions: Theory and Applications*; Imperial College Press: London, UK, 2004.
20. Arnold, B. Univariate and multivariate pareto models. *J. Stat. Distrib. Appl.* **2014**, *1*, 1–16. [CrossRef]
21. R Core Team. *A Language and Environment for Statistical Computing*; R Foundation for Statistical Computing: Vienna, Austria, 2023; ISBN 3-900051-07-0. Available online: http://www.R-project.org (accessed on 24 July 2024).
22. Dubey, S.D. Compound gamma, beta and F distributions. *Metrika* **1970**, *16*, 27–31. [CrossRef]
23. Bowman, K.O.; Shenton, L.R.; Gailey, P.C. Distribution of the ratio of gamma variates. Commun. *Stat. Simul. Comput.* **1998**, *27*, 1–19. [CrossRef]
24. Pham-Gia, T.; Duong, Q.P. The Generalized Beta- and F-distributions in statistical modelling. *Math. Comput. Model.* **1989**, *12*, 1613–1625. [CrossRef]
25. Crooks, G. *Field Guide to Continuous Probability Distributions. V1.0.0*; Berkeley Institute for Theoretical Sciences: Berkeley, CA, USA, 2006.
26. Tiao, G.; Guttman, I. The multivariate inverted beta distribution with applications. *J. Am. Stat. Assoc.* **1965**, *60*, 793–805. [CrossRef]
27. Cover, T.M.; Thomas, J.A. *Elements of Information Theory*; Wiley & Son, Inc.: New York, NY, USA, 2006.
28. Contreras-Reyes, J.E. Mutual information matrix based on Rényi entropy and application. *Nonlinear Dyn.* **2022**, *110*, 623–633. [CrossRef]
29. Tahmasebi, S.; Behboodian, J. Shannon entropy for the Feller-Pareto (FP) family and order statistics of FP subfamilies. *Appl. Math. Sci.* **2010**, *4*, 495–504.
30. Contreras-Reyes, J.E. Asymptotic form of the Kullback—Leibler divergence for multivariate asymmetric heavy-tailed distributions. *Physica A* **2014**, *395*, 200–208. [CrossRef]
31. Contreras-Reyes, J.E. Mutual information matrix based on asymmetric Shannon entropy for nonlinear interactions of time series. *Nonlinear Dyn.* **2021**, *104*, 3913–3924. [CrossRef]
32. Gradshteyn, I.; Ryzhik, I. *Table of Integrals, Series, and Products*, 7th ed.; Academic Press: New York, NY, USA, 2007.

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