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Analyzing the Selective Stock Price Index Using Fractionally Integrated and Heteroskedastic Models

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Abstract: Stock market indices are important tools to measure and compare stock market performance. The Selective Stock Price (SSP) index reflects fluctuations in a set value of financial instruments of Santiago de Chile's stock exchange. Stock indices also reflect volatility linked to high uncertainty or potential investment risk. However, economic shocks are altering volatility. Evidence of long memory in SSP time series also exists, which implies long-term persistence. In this paper, we studied the volatility of SSP time series from January 2010 to September 2023 using fractionally heteroskedastic models. We considered the Autoregressive Fractionally Integrated Moving Average (ARFIMA) process with Generalized Autoregressive Conditional Heteroskedasticity (GARCH) innovations—the ARFIMA-GARCH model—for SSP log returns, and the fractionally integrated GARCH, or FIGARCH model, was compared with a classical GARCH one. The results show that the ARFIMA-GARCH model performs best in terms of volatility fit and predictive quality. This model allows us to obtain a better understanding of the observed volatility and its behavior, which contributes to more effective investment risk management in the stock market. Moreover, the proposed model detects the influence volatility increments of the SSP index linked to external factors that impact the economic outlook, such as China's economic slowdown in 2012 and the subprime crisis in 2008.

Keywords: selective stock price; stock markets; volatility; GARCH model; long memory; ARFIMA model; FIGARCH model

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1. Introduction

Stock market indices are useful for measuring and comparing stock market performance, both in general and in specific economic sectors. These indices reflect the fluctuations in a set value of financial instruments that are part of these indices. Instruments include stocks, debt securities, and other types of assets. In the study of stock market indices, intrinsic value and volatility of the index carry key importance. Moreover, understanding volatility indicators of financial variables is essential for adequate risk management both for market participants and regulators (Alfaro and Silva 2008). Volatility is a fundamental concept in financial sciences that reflects uncertainty or risk associated with investments, with high (low) volatility indicating high (low) risk (Andersen and Teräsvirta 2009). On the other hand, long memory is a key concept for modeling volatility or heteroskedasticity in stock markets or other macroeconomic indices in time series analysis (Ding et al. 1993; Palma 2007), but econometric applications have not considered long memory and volatility dependency.

The main stock market index of Santiago de Chile's stock exchange is the Selective Stock Price (SSP) index[1]. This index represents the performance of 40 stocks with the highest trading volume and liquidity on the Chilean stock market and helps in analyzing the general behavior of the market. Yet, SSP index literature is limited, with one highlight being a Central Bank study Alfaro and Silva (2008), which analyzes SSP volatility using a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model (Bollerslev 1986). The results showed that during periods of turbulence or high volatility, the historic parameters of the SSP index became less relevant, which renders short-term information more important. However, it is necessary to consider that the correlation of SSP returns is positive, like the stock indices that make up the Latin American integrated market (Aragón 2017).

1.1. Literature Review

Stock market time series often present non-linear behavior. Strong evidence exists regarding a large number of financial time series that shows stylized facts, such as clusters of observations (original time series or their squares) with high variability followed by clusters with low variability but strong autocorrelation (Palma 2016). Hence, it is necessary to use heterokedastic models when the conditional variance of past observations is not constant.

Given the importance of modeling the volatility of financial time series, Engle (1982) proposed an Autoregressive Conditional Heteroskedasticity (ARCH) econometric model to analyze high-volatility, non-stationary time series data, i.e., data where error variance changes over time. The ARCH model assumes that variance is conditioned on past observations but does not consider the influence of past volatility. Bollerslev (1986) extended the ARCH model to a GARCH one. In addition to square observations, the GARCH model also includes the past values of conditioned variance, allowing researchers to capture the influence of conditioned volatility of past periods in the estimation of conditional variance of a current period. Alfaro and Silva (2008) examined some volatility measures of the SSP index using the GARCH model, among others, and found that the historical parameters of SSP time series became less relevant during high-volatility periods, providing more short-term information. In addition, when intra-day information was incorporated, estimated volatility was more efficient than that considering only the closing price of the stock.

Yet, long memory in the time series context is crucial. This concept refers to long-term persistence in time series data, with past observations impacting future ones (Granger and Joyeux 1980; Hosking 1981). Long-term persistence manifests as autocorrelation with slow decaying, persistent fluctuations, or prolonged heteroskedastic variations. Long memory has been observed in several scientific fields, such as hydrology, geophysics, and economics (Box et al. 2015; Contreras-Reyes 2022; Contreras-Reyes and Palma 2013). With respect to financial economics and stock market indices, certain asset performance in speculative markets was approximately uncorrelated but not independent over time (Baillie et al. 1996b). The world's main stock market indices—the S&P 500 (New York), EOE (Amsterdam), DAX (Frankfurt), and Hang Seng (Hong Kong)—tend to reject the unit root test, indicating that these time series are non-stationary and, therefore, a search for alternatives such as fractionally integrated models is required (Gil-Alana 2006). Fractional integration allows researchers to address the non-stationarity of time series, enables capturing of the long-memory presence of the process, and reduces bias in predictions. More recently, Khumalo et al. (2023) considered other extensions of the GARCH model (FIAPARCH and HYGARCH) and the mentioned FIGARCH model to quantify the Johannesburg stock market value at risk.

Another way to model long memory involves Autoregressive Fractionally Integrated Moving Average (ARFIMA) models (Contreras-Reyes and Palma 2013; Palma 2007), which estimate the mean of the process using the short-term memory through an Autoregressive Moving Average (ARMA) structure and the long-term memory through fractionally integrated differencing. Nevertheless, Baillie et al. (1996b) introduced an alternative approach to modeling long memory, involving the Fractionally Integrated Generalized Auto-Regressive Conditionally Heteroskedastic (FIGARCH) model. In contrast to the ARFIMA model used for the mean of the process, the FIGARCH model captures dependency of process volatility through innovations. This approach is useful for modeling long-term dependencies in absolute and square returns of financial assets. Bollerslev and Mikkelsen (1996) analyzed several financial assets of US stock markets, using fractionally exponential GARCH models to characterize long-term dependencies in their volatility. They found that the S&P500 index produced an optimal fit to a fractionally integrated process with mean reverse, such as the

FIGARCH model. This finding highlighted that considering the long-term persistence and regression mean trend in modeling financial asset volatilities is important.

Another way to model long memory and heteroskedasticity in time series involves an ARFIMA-GARCH model (Baillie et al. 1996a), which combines an ARFIMA model for the process and a GARCH one for the innovations. This approach is useful to analyze the relationships between the conditional mean and variance of the long-memory process, the slow decay of autocorrelations, and heteroskedasticity. In addition, Baillie et al. (1996a) studied inflation in 10 countries, modeling this index using an ARFIMA-GARCH model and the estimated volatility via quasi-maximum log-likelihood estimation (QMLE) and based on Student-t innovations. The results showed that shocks or inflation changes include long memory and persistence over time. It was also observed that changes are not permanent, despite long-term influence, as inflation tends to stabilize around a more stable average.

1.2. Study Objectives

Given that the GARCH model proposed by Alfaro and Silva (2008) does not include fractional parameters, unlike FIGARCH or ARFIMA-GARCH, it is limited to simultaneously modeling the long-memory and volatility properties in the SSP index. Considering the importance of stock market indices, the motivation for this study is to create an SSP volatility forecasting method and obtain a tool that allows for better control of market risk through implicit modeling of the main index of Chile's stock market. The main objective of this study is to compare the efficiency of fractional models (ARFIMA-GARCH and FIGARCH) in terms of volatility fit performance and predictive quality related to the SSP index with a classical GARCH one.

Considering the foundational theories of the FIGARCH model provided by Bollerslev and Mikkelsen (1996) and the extension of the GARCH model given by the ARFIMA-GARCH model developed by Baillie et al. (1996a), we fitted an SSP log returns time series (January 2010–September 2023) with the GARCH, ARFIMA-GARCH, and FIGARCH models using the QMLE method and evaluated the fit quality of these models using the Akaike (AIC) and Bayesian (BIC) information criteria (see, e.g., Chávez et al. 2023) and residual diagnostic methods using the autocorrelation function (ACF) (see, e.g., Contreras-Reyes and Palma 2013) and the weighted Ljung–Box test (Fisher and Gallagher 2012). Subsequently, we evaluated the predictive quality of the best model through cross validation to estimate implicit SSP volatility.

This paper is organized as follows. Section 2 presents a description of the SSP index data and log returns. Section 3 presents the methodology, including the GARCH, ARFIMA-GARCH, and FIGARCH models, information criteria, residual diagnostic methods, and Student-t distribution for innovations. Sections 4 and 5 present the main results and volatility estimation, respectively. Finally, the discussion and conclusions are presented in Section 6.

2. Selective Stock Price Index

In financial sciences, time series are provided by indices and asset price evolution (Nelson 1991). A common topic is the asset price evolution of the SSP index. Nevertheless, several studies have focused on asset returns instead of asset prices. Campbell et al. (1998) and Tsay (2005) argued for the use of returns, as they reflect the full summary and freedom of scales about a possible investment opportunity and because return time series are easier to handle than asset time due to their more adequate statistical properties. While many ways to define asset performance exist, a simplified return definition is given by the following:

$$\frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}, \tag{1}$$

where P_t is the index price at time t. However, logarithmic transformation of the original time series is often used in stock market index analysis to stabilize process variance. In addition, differentiating the logarithmic term stabilizes the time series mean to obtain the log returns given by the following:

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right) = \log(P_t) - \log(P_{t-1}). \quad (2)$$

We assigned the SSP index to P_t. SSP value was determined based on capital variations in each asset of this index using the relative weight of each one. Major companies that are part of this index include SQM-B, Banco de Chile, Santander, and Copec. The closing prices of the SSP index were obtained from Central Bank of Chile webpage (https://www.bcentral.cl/, accessed on 1 April 2023). Figure 1 illustrates the SSP index between 2 January 2010 and 30 September 2023. Crucially, the SSP index is provided for working days only, but if missing values were estimated using interpolation techniques, an autocorrelation of lag 7 (days) and related to weekends emerged. Hence, this study excluded the missing days from the SSP time series to avoid bias in model fits. The Augmented Dickey–Fuller test (Cheung and Lai 1995) confirmed that $\log(P_t)$ is non-stationary, with a Dickey–Fuller statistic of -2.6054 (with a lag order of 14 and a p value of 0.322); i.e., we cannot reject the null hypothesis of non-stationarity because the p value is not smaller than 0.05. Therefore, differentiation (2) is required for the next analysis. SSP log returns are also presented in Figure 1, where the COVID-19 pandemic period is highlighted. However, other crises appear in log-returns, such as China's economic slowdown in 2012 and the subprime crisis in 2008.

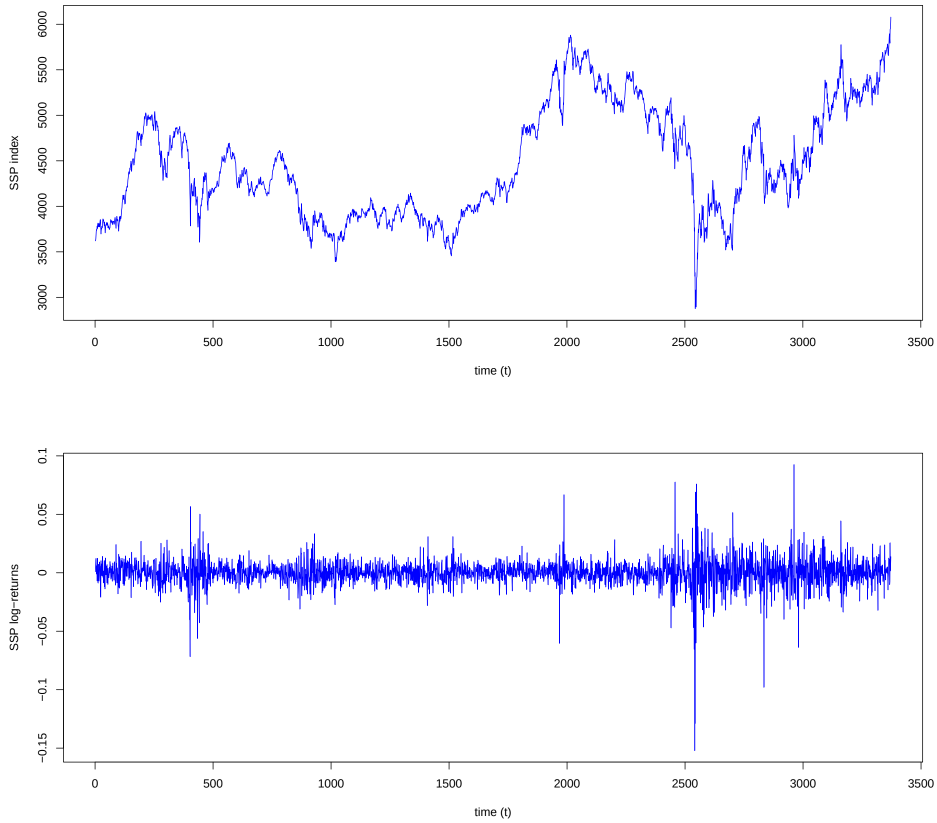

Figure 1. Selective Stock Price index (**top**) and log returns (**bottom**), 2 January 2010 to 30 September 2023.

An analysis of long-memory presence using the sample ACF provided in Figure 2, also for absolute ($|r_t|$) and square (r_t^2) values of SSP log-returns, follows. The sample ACF provides autocorrelations close to zero or close to Bartlett's bands (95% confidence level), suggesting that log returns are uncorrelated. However, considering the absolute and square log returns, both plots provide relevant information for volatility modeling, because GARCH models could not capture long-memory dependency in correlations. Still, both plots that sample ACF values exceeded Bartlett's bands at the 95% confidence level, even though the values were concentrated between 0.2 and 0.4, which was relatively low. Additionally, the sample ACF is only a graphical tool for absolute and square SSP log returns and does not guarantee significance of the fractional differencing parameter of fractional GARCH models.

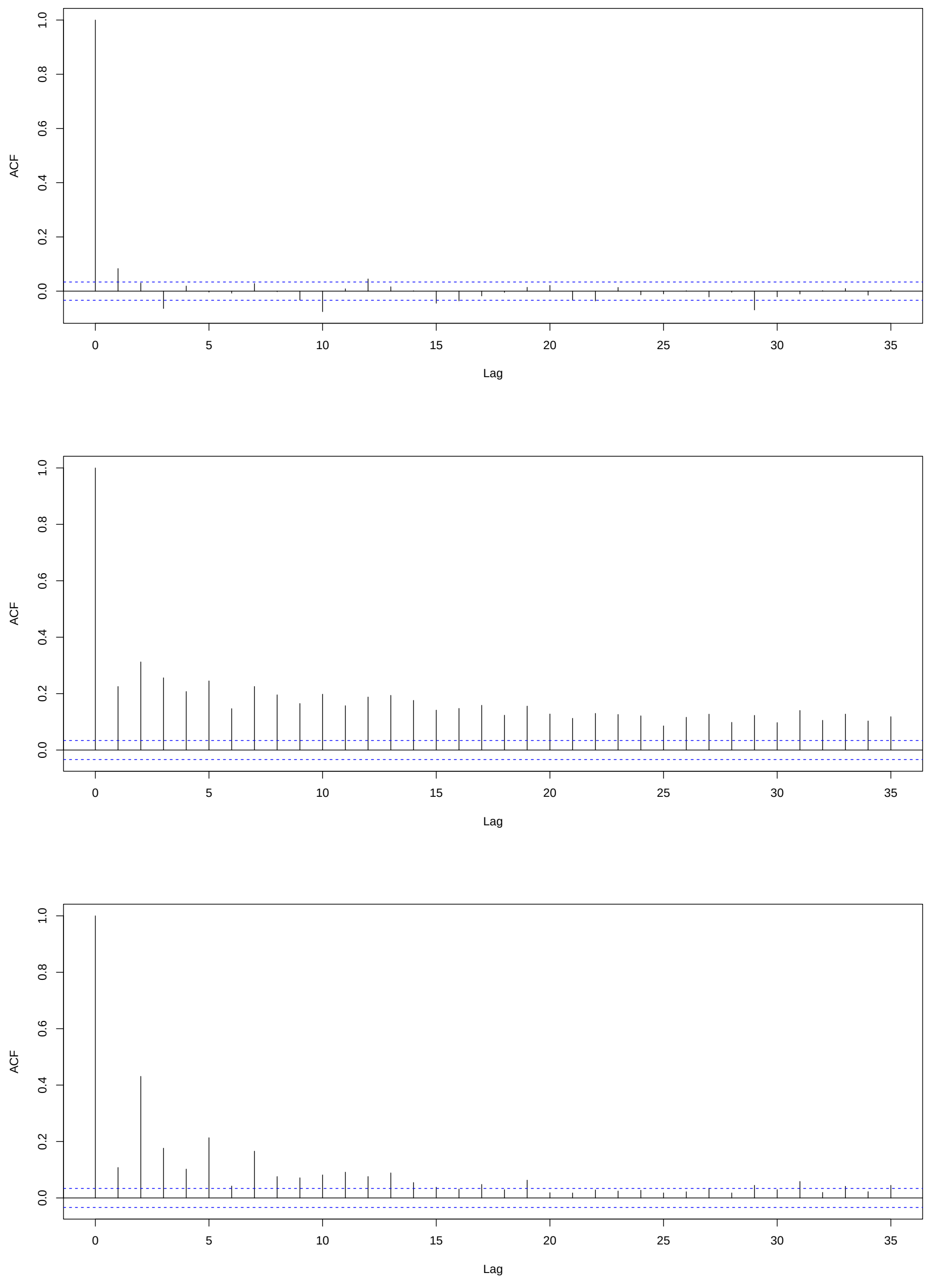

Figure 2. Sample autocorrelation function of SSP log-returns (**top**). Absolute value of SSP log returns (**middle**) and squares of SSP log returns (**bottom**).

Distribution associated with innovations is also key for financial time series modeling. Palma (2016) found that financial time series tend to exhibit heavy tails and excess of kurtosis (more kurtosis than Gaussian distribution) in innovations. This phenomenon is observed in SSP log returns, as illustrated in Figure 3, which shows an excess of leptokurtosis. Therefore, a Student-t distribution could be used for GARCH innovation modeling (Baillie et al. 1996b; Chávez et al. 2023), as a Student-t distribution provides more flexibility for heavy-tailed data fit, commonly observed in financial time series. This approach is necessary to obtain estimated parameters with smaller bias and more precise predictions for SSP volatility.

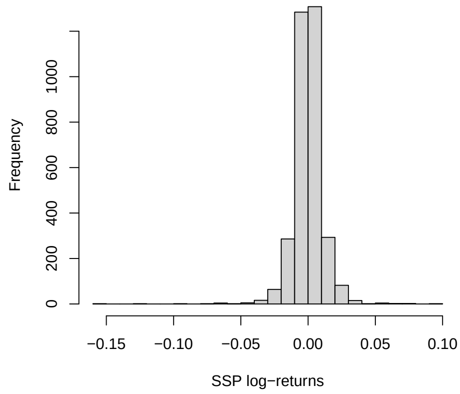

Figure 3. Histogram of SSP log returns (2 January 2010 to 30 September 2023).

3. Methods

3.1. ARMA Model

A process $\{y_t\}$, $t \in \mathbb{Z}$ follows an ARMA model of autoregressive and moving-average orders p and q, respectively, as follows:

$$\phi(B)y_t = \theta(B)\epsilon_t \tag{3}$$

where $\phi(B) = 1 + \phi_1 B + \ldots + \phi_p B^p$ and $\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q$ are the autoregressive and moving-average polynomial operators, respectively, and B is the backshift operator. In addition, $\phi(B)$ and $\theta(B)$ have no common roots (Box et al. 2015). Innovations of model (3) are assumed white noise, $\{\epsilon_t\} \sim RB(0, \sigma^2)$. The model (3) can be written as follows:

$$\left(1 - \sum_{i=1}^{p} \phi_i B^i\right) y_t = \left(1 + \sum_{j=1}^{q} \theta_j B^j\right) \epsilon_t, \tag{4}$$

and is denoted $y_t \sim ARMA(p, q)$. ARMA models are based on stationarity conditions in the mean, variance, and autocovariance, as the mean and variance are finite and constant with respect to t. The autocovariance function does not depend on time; only on the number of lagged periods (lags).

3.2. ARFIMA Model

ARFIMA is a long-memory class model that accounts explicitly for the persistence and long-term correlations of time series. The general expression for this model denotes $y_t \sim \text{ARFIMA}(p, d, q)$ and is given by the following:

$$\phi(B)y_t = \theta(B)(1 - B)^{-d}\epsilon_t, \tag{5}$$

where $\phi(B)$ and $\theta(B)$ are defined in the latter section. $d \in (-1, 1/2)$ is the fractional differencing operator given by binomial expansion

$$(1 - B)^{-d} = \sum_{j=0}^{\infty} \eta_j B^j = \eta(B), \tag{6}$$

with

$$\eta_j = \frac{\Gamma(j + d)}{\Gamma(j + 1)\Gamma(d)},$$

and $\Gamma(\cdot)$ is the common gamma function. If polynomials $\phi(B)$ and $\theta(B)$ have no common roots and $d \in (-1, \frac{1}{2})$, the stationarity, causality, and invertibility of the process can be established (Contreras-Reyes and Palma 2013). The estimation method for ARFIMA parameters is based on the Whittle estimator (Contreras-Reyes and Palma 2013).

3.3. GARCH Model

The ARCH model considers that variance in time series can change and is conditioned by past observations. An ARCH model of order r, denoted as $ARCH(r)$, is defined at discrete time t by the following:

$$\begin{aligned} y_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{r} \alpha_i y_{t-i}^2, \end{aligned} \tag{7}$$

where $\sigma_t^2 = \mathbb{E}[y_t^2 | y_{t-1}, y_{t-2}, \ldots]$ is the conditional variance of process $\{y_t\}$ and denotes the expected value of y_t^2, conditioned to the set of past observations $\{y_{t-1}, y_{t-2}, \ldots\}$. In addition, α_0 and α_i, $i = 1, \ldots, r$ are positive real values, and condition $\sum_{i=1}^{r} \alpha_i < 1$ ensures the stationarity of process $\{y_t\}$. $\{\epsilon_t\}$ is a sequence of random variables that are independent and identically distributed (i.i.d.); i.e., white noise.

On the other hand, the GARCH model also considers past observations modeled by autoregressive parameters. However, it also includes the dependency of past variance values, which implies the presence of heteroskedasticity as a function of observed time. Therefore, a GARCH model is an extension of an ARCH one, given as follows:

$$\begin{aligned} y_t &= \sigma_t \epsilon_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{j=1}^{r} \alpha_j y_{t-j}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2, \end{aligned} \tag{8}$$

where in the ARCH case (Engle 1982), $\sigma_t^2 = \mathbb{E}[y_t^2 | y_{t-1}, y_{t-2}, \ldots]$ is the conditional variance of process $\{y_t\}$. The coefficients of the GARCH model, $\alpha_0, \alpha_1, \ldots, \alpha_r$ and $\beta_0, \beta_1, \ldots, \beta_s$, are positive real numbers, and $\{\epsilon_t\}$ is an i.i.d. sequence of random variables. Under stationary conditions, the parameters of the GARCH process accomplish $\sum_{j=1}^{r} \alpha_j + \sum_{j=1}^{s} \beta_j < 1$.

Despite being an effective tool to model heteroskedasticity in a financial time series, the GARCH model does not address the possible existence of long-term dependency in the process.

3.4. ARFIMA-GARCH Model

This approach combines long-memory process modeling using an ARFIMA model and heteroskedastic modeling with a GARCH one. The process is defined at a discrete time, denoted as ARFIMA(p, d, q)-GARCH(r, s) and is given as follows:

$$\begin{aligned} \phi(B) y_t &= \theta(B)(1 - B)^{-d} \epsilon_t, \\ \epsilon_t &= e_t \sigma_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{j=1}^{r} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2, \end{aligned}$$

where $\sigma_t^2 = \mathbb{E}[\epsilon_t^2 | \epsilon_{t-1}, \epsilon_{t-2}, \ldots]$ is the conditional variance of process $\{\epsilon_t\}$. The coefficients of a GARCH model, $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$, are positive real values, such that $\sum_{j=1}^{r} \alpha_j + \sum_{j=1}^{s} \beta_j < 1$ (for stationarity of process ϵ_t), and $\{e_t\}$ is an i.i.d. sequence of random variables with zero mean and unit variance (Baillie et al. 1996b; Ramŕez-Parietti et al. 2021).

3.5. FIGARCH Model

Given that squares of financial time series present similar or higher-than-return autocorrelations, the reduction in memory that affects the squares of the ARFIMA-GARCH processes might not work, implying other approaches need to be studied to directly model the strong dependency of return squares (Palma 2007). One option is the FIGARCH model, which proved efficient for several financial assets such as stocks, stock indices, and exchange rates. The model allows capturing of the correlation persistence in volatility, making it a useful tool to understand financial risk.

To define the FIGARCH model, first consider the variance definition of the GARCH model given in (8), which allows us to define the following polynomials:

$$\boldsymbol{\alpha}(B) = \sum_{j=1}^{p} \alpha_j B^j \quad \text{y} \quad \boldsymbol{\beta}(B) = \sum_{j=1}^{q} \beta_j B^j.$$

Considering the latter polynomials, an initial model for y_t is proposed, given as follows:

$$[1 - \boldsymbol{\alpha}(B) - \boldsymbol{\beta}(B)]y_t^2 = \alpha_0 + [1 - \boldsymbol{\beta}(B)]\nu_t, \tag{9}$$

where $\nu_t = y_t^2 - \sigma_t^2$. On the left side of Equation (9), the model is rewritten as an integrated autoregressive one as follows:

$$(1 - B)\boldsymbol{\gamma}(B)y_t^2 = \alpha_0 + [1 - \boldsymbol{\beta}(B)]\nu_t, \tag{10}$$

where

$$\boldsymbol{\gamma}(B) = \sum_{j=1}^{m-1} \gamma_j B^j,$$

and $m = \max\{p, q\}$. Note in (10), a whole differentiation of lag 1 is considered. Model (10) is defined as IGARCH(r, s).

Finally, a FIGARCH(r, d, s) model developed by Baillie et al. (1996b) is proposed, which includes a fractionary parameter d by replacing term $(1 - B)$ with $(1 - B)^d$ in (10) to obtain

$$\boldsymbol{\gamma}(B)y_t^2 = (1 - B)^{-d}(\alpha_0 + [1 - \boldsymbol{\beta}(B)]\nu_t), \tag{11}$$

where $(1 - B)^{-d}$ is obtained in (6).

In addition, a common parameter estimation method for heteroskedastic models is the QMLE option (Baillie et al. 1996b). The log-likelihood function of a heteroskedastic model, assuming that innovations are normally distributed, is as follows:

$$\mathcal{L}(\boldsymbol{\Omega}) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left[\log(\sigma_t^2) + \frac{\epsilon_t^2}{\sigma_t^2}\right], \tag{12}$$

where n is the number of observations and $\boldsymbol{\Omega} = (d, \alpha_0, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$ for FIGARCH. Therefore, QMLE $\hat{\boldsymbol{\Omega}}$ is obtained by maximizing Equation (12). It is possible to modify σ_t^2 based on the GARCH variance of (8) for parameter set $\boldsymbol{\Omega} = (\alpha_0, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_p)$.

In several financial time series, the innovations of the heteroskedastic models could be assumed as leptokurtic and standardized (Chávez et al. 2023; Ghalanos 2023). In this sense, the GARCH model with Student-t innovations was used for the first time by Bollerslev (1987) as an alternative to Gaussian distribution. The location μ, scale τ, and heavy-tails parameters were considered for Student-t and directly modeled based on the degree of freedom parameter ν in the following form:

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\tau\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{(x-\mu)^2}{\tau\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}. \tag{13}$$

It is possible to obtain the standardized version via substitution $z = (x - \mu)/\tau$ in (13). Student-t is a unimodal and symmetric distribution, where the first moment of X is μ, and the variance is as follows:

$$Var(x) = \tau\frac{\nu}{(\nu - 2)}, \tag{14}$$

with $\nu > 2$.

3.6. Model Selection

In this section, we consider two criteria commonly used for model selection (see, e.g., Chávez et al. 2023). The first is AIC, which is defined over a set of K competitor model indices by k, $\{M_k\}$, $k = 1, 2, \ldots, K$ and given by the following:

$$\text{AIC}(M_K) = -2\mathcal{L}(\widehat{\boldsymbol{\Omega}}_k) + 2g, \tag{15}$$

where g is the number of parameters of model M_k, $\mathcal{L}(\widehat{\boldsymbol{\Omega}}_k)$ is the associated log-likelihood function of the kth model, and $\widehat{\boldsymbol{\Omega}}_k$ is the set of estimated parameters. The best model is the one with the lowest AIC.

The second criterion is BIC, which is also defined for set $\{M_k\}$ as follows:

$$\text{BIC}(M_k) = -2\mathcal{L}(\widehat{\boldsymbol{\Omega}}_k) + g\log(n). \tag{16}$$

The best model is the one with the lowest BIC, as it indicates the optimal balance between the goodness of fit in data and the number of parameters (model complexity). For sample size $n > 8$, BIC strongly penalizes the increment of the number of parameters in the model compared with AIC.

4. Results

Here, we evaluate the models and determine if fractionally integrated heteroskedastic models provide a better fit of volatility of the SSP index compared with non-fractional models. We considered the log-returns of the closing prices of the SSP index described in Section 2. The estimation, diagnostic, and prediction procedures of the models were carried out using `R` software (R Core Team 2023), using the `Rugarch` package (Ghalanos 2023).

4.1. GARCH Model

A first step was the identification of the optimal order for the GARCH model to be fitted onto SSP time series, which involved comparing AIC and BIC values (Table 1), with the GARCH$(1,1)$ model emerging as the most appropriate one.

Table 1. AIC and BIC of GARCH models fitted to SSP time series and for several orders.

r	s	AIC	BIC
1	1	−6.5820	−6.5648
1	2	−6.5817	−6.5628
1	3	−6.5828	−6.5622
2	1	−6.5813	−6.5624
2	2	−6.5811	−6.5604
2	3	−6.5828	−6.5603
3	1	−6.5806	−6.5600
3	2	−6.5804	−6.5580
3	3	−6.5822	−6.5580

Considering the optimal GARCH$(1,1)$ model, the estimated parameters appear in Table 2. All of the estimated parameters were significant, and $\alpha_1 + \beta_1 = 0.983 < 1$, fulfilling the stationarity requirement. Additionally, $\widehat{\nu}6$, which was relatively low (<10), indicated that the innovations were fitted by Student-t with heavy tails (as observed in Figure 3, where the SSP log returns clearly present heavy tails). This result indicated that it was a better fit of innovations with a Student-t distribution instead of a Gaussian one.

Table 2. Estimated parameters of GARCH(1, 1) model for SSP log returns.

	Estimates	Std. Error	*t* Value	*p* Value
α_0	0.002	0.003	0.643	0.520
α_1	0.099	0.048	2.080	0.038
β_1	0.885	0.049	17.901	<0.01
ν	5.936	0.177	33.499	<0.01

The diagnostic analysis for the selected GARCH model innovations involved a histogram of residuals (Figure 4), which show leptokurtic behavior, suggesting the Student-*t* distribution provided a better fit than the Gaussian one. Moreover, the sample ACF indicated small autocorrelations (close to Bartlett's bands of the 95% confidence level), validating the white noise hypothesis. But, the sample ACF exceeded the Bartlett band limits for square residuals. Table 3 illustrates the weighted Ljung–Box test results, where for all lags the white noise hypothesis were rejected at the 95% confidence level. On the contrary, no evidence exists for rejecting the white noise hypothesis for square residuals.

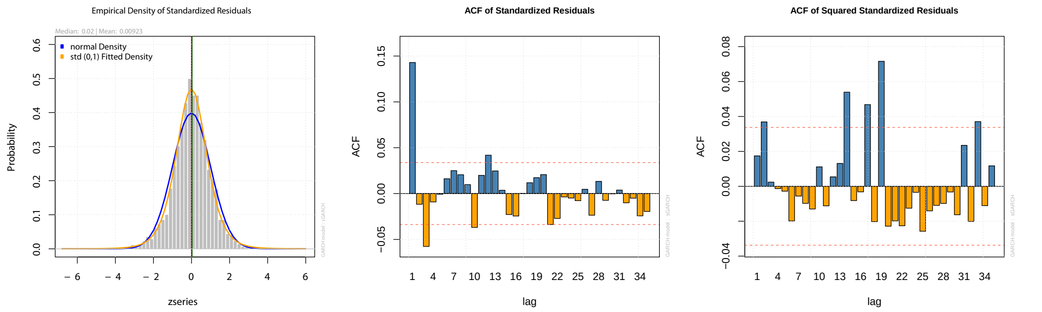

Figure 4. Left to right: Histogram of standardized residuals and sample ACF of residuals and square residuals of the GARCH(1, 1) model.

Table 3. Weighted Ljung–Box test for GARCH(1, 1) model residuals.

Standardized Residuals			Squared Standardized Residuals		
Lag	Statistic	*p* Value	Lag	Statistic	*p* Value
1	68.98	<0.01	1	1.025	0.3113
8	69.21	<0.01	5	4.718	0.1771
14	76.19	<0.01	9	5.904	0.3094

4.2. ARFIMA-GARCH Model

For this model, the AIC and BIC were applied to find the ARFIMA and GARCH parts related to SSP log returns. For ARFIMA, an ARMA(2, 1) order and a fractionally integrated parameter $\widehat{d} = 0.064$ were found. For GARCH, a GARCH(1, 1) order was found. Therefore, the resulting ARFIMA(2, d, 1)-GARCH(1, 1) was selected as the best model based on an AIC and BIC of -0.65 and -6.63, respectively. Table 4 shows the estimated parameters of the model, where the estimated parameters ϕ_1, ϕ_2, and θ_1 of ARFIMA were significant, but the fractionally integrated parameter was not significant at the 95% confidence level. For GARCH, the estimated α_1 and β_1 parameters were significant, associated with the heterokedastic components of SSP log returns. Also, $\alpha_1 + \beta_1 = 0.984 < 1$, which accomplished the stationarity requirement. Additionally, the estimated ν parameter was significant and close to 6, indicating that the residuals were modeled assuming a heavy-tailed distribution.

Table 4. Estimated parameters of ARFIMA$(2, d, 1)$-GARCH$(1, 1)$ model for SSP log returns.

	Estimates	Std. Error	*t* Value	*p* Value
ϕ_1	0.739	0.085	8.688	<0.01
ϕ_2	−0.135	0.019	−7.266	<0.01
θ_1	−0.651	0.110	−5.900	<0.01
d	0.065	0.052	1.243	0.266
α_0	0.002	0.002	0.797	0.425
α_1	0.099	0.037	2.650	0.008
β_1	0.885	0.039	22.837	<0.01
ν	5.972	0.337	17.726	<0.01

With respect to diagnostic analysis of the optimal model, a histogram plot was created (Figure 5), showing leptokurtosis and suggesting that Student-*t* density was more adequate than the normal one. In addition, Bartlett's bands of sample ACF for absolute and square standardized residuals showed that the residuals were uncorrelated at the 95% confidence level[2] This result showed that the proposed model fits the SSP log returns considering volatility shocks over time. Evaluating the residuals with a weighted Ljung–Box test (Table 5) confirmed that the residuals were white noise.

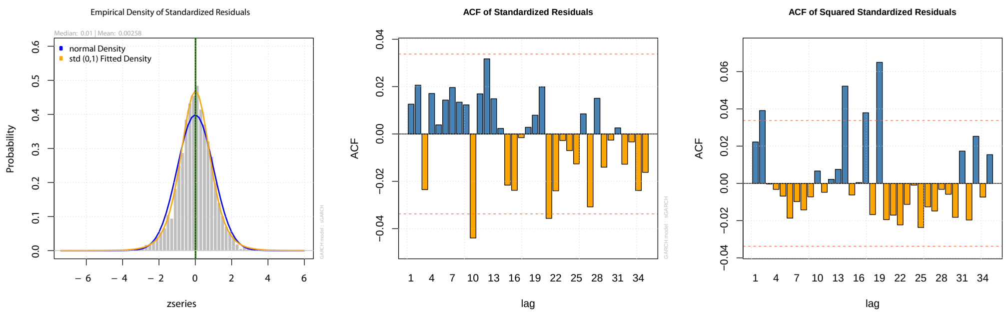

Figure 5. Left to right: Histogram of standardized residuals, sample ACF of residuals, square residuals of ARFIMA$(2, d, 1)$-GARCH$(1, 1)$ model.

Table 5. Weighted Ljung–Box test for ARFIMA$(2, d, 1)$-GARCH$(1, 1)$ model residuals.

Standardized Residuals			Squared Standardized Residuals		
Lag	**Statistic**	***p* Value**	**Lag**	**Statistic**	***p* Value**
1	0.535	0.464	1	1.668	0.197
8	4.495	0.485	5	5.844	0.098
14	9.444	0.171	9	7.712	0.185

4.3. FIGARCH Model

AIC and BIC were applied to obtain the optimal FIGARCH order, which was a FIGARCH $(1, d, 1)$ (with AIC of −6.64 and BIC of −6.629). In contrast to ARFIMA-GARCH, the differencing fractionary operator is considered for model residuals, avoiding a specification of an ARMA order. The estimation results can be seen in Table 6, where the estimated fractionally integrated parameter was significant and close to 0.1. Additionally, $\alpha_1 + \beta_1 = 0.248 < 1$, which accomplished the stationarity requirement. The estimated ν parameter was significant and close to 6, again indicating the presence of a heavy tail in the residual distribution.

Table 6. Estimated parameters of FIGARCH$(1, d, 1)$ model for SSP log returns.

	Estimates	Std. Error	t Value	p Value
α_0	0.007	0.003	2.332	0.020
α_1	0.031	0.134	0.231	0.817
β_1	0.217	0.156	1.390	0.165
d	0.083	0.016	5.296	<0.01
ν	6.024	0.590	10.202	<0.01

Figure 6 shows the histogram of residuals with normal and Student-t fit. A leptokurtosis is notable, suggesting that residuals are better fitted by Student-t density. With respect to sample ACFs, the values were higher than Bartlett's bands for several lags, indicating significant autocorrelations of residuals. The presence of autocorrelation in the residuals suggests the existence of subjacent patterns in SSP log returns, which are still not explained by a FIGARCH model, affecting inferential validity. For validation, we considered the weighted Ljung–Box test in Table 7, which revealed that for lags 1, 2, and 5, the residuals were white noise. Thus, the null hypothesis was rejected. A reason could be related to periodic volatility increments due to the crisis. On the other hand, for lags 1, 5, and 9, the squared standardized residuals were white noise, validating that the FIGARCH model considered the square SSP log returns and square residuals.

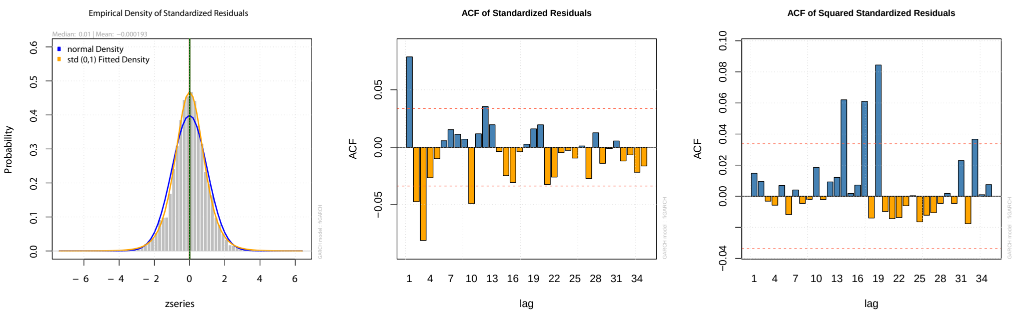

Figure 6. Left to right: Histogram of standardized residuals, sample ACF of residuals, square residuals of FIGARCH$(1, d, 1)$ model.

Table 7. Weighted Ljung–Box test for FIGARCH$(1, d, 1)$ model residuals.

Standardized Residuals			Squared Standardized Residuals		
Lag	**Statistic**	**p Value**	**Lag**	**Statistic**	**p Value**
1	20.90	<0.01	1	0.736	0.391
2	24.69	<0.01	5	1.073	0.843
5	41.34	<0.01	9	1.438	0.961

5. Volatility Estimation

The heteroskedastic models considered in the previous section do not provide an explicit formula for volatility. In this section, we show that conditional volatility is an implicit function at time t. We assume that an investor distributes wealth among a risk-free asset (e.g., a bonus) and a risk asset that replicates the SSP index as follows:

$$w_t = a_t + b_t, \tag{17}$$

where w_t, a_t and b_t represent the wealth, risk asset, and risk-free asset at time t. That investor is assumed to gain wealth at time $t+1$:

$$w_{t+1} = (1 + r_{t+1})a_t + (1 + r^f_{t+1})b_t, \tag{18}$$

where r_{t+1} is the efficiency of the SSP index (or log return), and r^f_{t+1} is the risk-free bonus income at time $t+1$. By substituting $b_t = w_t - a_t$ from (17) into (18), we obtain the following:

$$\begin{aligned} w_{t+1} &= (1+r_{t+1})a_t + (1+r^f_{t+1})(w_t - a_t), \\ &= (r_{t+1} - r^f_{t+1})a_t + (1+r^f_{t+1})w_t, \end{aligned} \tag{19}$$

where the term $r_{t+1} - r^f_{t+1}$ is known as the return excess, risk premium, or required spread by the investor. Then, the investor is interested in finding the optimal a_t to maximize usefulness or benefits as follows:

$$\max_{\{a_t\}} \left\{ u(w_t) + \lambda \mathbb{E}_t[u(w_{t+1})] \right\}, \tag{20}$$

subject to (19), where $u(w_t)$ is the investor benefit function that depends on wealth w_t, and $\mathbb{E}_t[\cdot]$ represents the conditional expected value of the available information at time t. In addition, λ is the investor's discount factor. By replacing (19) in (20), we obtain the following:

$$\max_{\{a_t\}} \left\{ u(w_t) + \lambda \mathbb{E}_t[u((r_{t+1} - r^f_{t+1})a_t + (1+r^f_{t+1})w_t)] \right\}. \tag{21}$$

By computing the first derivatives with respect to a_t and equaling them to 0, we obtain the following:

$$\mathbb{E}_t\left[\lambda \frac{\partial u(w_{t+1})}{\partial a_t}(r_{t+1} - r^f_{t+1})\right] = 0,$$

which is equivalent to the following:

$$\frac{\partial u(w_t)}{\partial a_t}\mathbb{E}_t\left[u_\lambda(w_{t+1})(r_{t+1} - r^f_{t+1})\right] = 0.$$

Given that $\frac{\partial u(w_t)}{\partial a_t} \neq 0$, the expected value term is equal to 0, and the term

$$u_\lambda(w_{t+1}) = \lambda \frac{\frac{\partial u(w_{t+1})}{\partial a_t}}{\frac{\partial u(w_t)}{\partial a_t}},$$

is known as the stochastic discount factor. Under the definition of covariance function, we obtain the following:

$$\mathbb{E}_t[u_\lambda(w_{t+1})(r_{t+1} - r^f_{t+1})] = Cov\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right) + \mathbb{E}_t[u_\lambda(w_{t+1})]\mathbb{E}_t\left[r_{t+1} - r^f_{t+1}\right].$$

And by using the definition of the usual correlation between two random variables, we obtain the following:

$$\begin{aligned} \mathbb{E}_t\left[r_{t+1} - r^f_{t+1}\right] &= \frac{-Cov\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right)}{\mathbb{E}_t[u_\lambda(w_{t+1})]} \\ &= \frac{-Cov\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right)}{\sqrt{Var(u_\lambda(w_{t+1}))Var(r_{t+1} - r^f_{t+1})}} \frac{\sqrt{Var(u_\lambda(w_{t+1}))Var(r_{t+1} - r^f_{t+1})}}{\mathbb{E}_t[u_\lambda(w_{t+1})]}, \\ &= -Corr\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right) \frac{\sqrt{Var(u_\lambda(w_{t+1}))Var(r_{t+1} - r^f_{t+1})}}{\mathbb{E}_t[u_\lambda(w_{t+1})]}, \end{aligned}$$

where $Var(x)$ denotes the variance of random variable x. For simplicity, we assume that r^f_{t+1} is constant in the last expression to obtain the following:

$$Var(r_{t+1}) = \frac{\mathbb{E}_t\left[r_{t+1} - r^f_{t+1}\right]^2}{Corr\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right)^2}\left(\frac{Var(u_\lambda(w_{t+1}))}{\mathbb{E}_t[u_\lambda(w_{t+1})]}\right)^{-2}.$$

In the last expression, we observe that the return variance of the risk asset that replicates the SSP index depends on three market and structural factors associated with investor preferences in Chile:

- $\mathbb{E}_t\left[r_{t+1} - r^f_{t+1}\right]^2$: market risk premium required by the investor, which varies according to macroeconomic shocks or crises such COVID-19 or the 2019 social protests;
- $Corr\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right)^2$: squared correlation between asset return and investor wealth, which depends on investor preferences;
- $\frac{Var(u_\lambda(w_{t+1}))}{\mathbb{E}_t[u_\lambda(w_{t+1})]}$: variation coefficient or market risk assessment associated with investor risk perception.

The SSP log-return volatility (and persistence) modeled using a GARCH, ARFIMA-GARCH, or FIGARCH model depends on the persistence of structural factors related to preferences of Chilean stock market investors. However, the GARCH, ARFIMA-GARCH, and FIGARCH models do not explicity describe the conditional volatility of SSP log returns.

In financial time series, variations in stock values normally exist. Obtaining the real (observed) volatility of the process is complex, but it is possible to identify the volatility structure in some periods with the help of a heteroskedastic model (Gonzalez-Rivera et al. 2004). In this study, we considered the GARCH, ARFIMA-GARCH, and FIGARCH models, which explain conditional variability over time, recognizing that temporal volatility may change in response to past events (Palma 2016). Figure 7 presents the estimated volatility ($\widehat{\sigma}^2_t$) for these models from January 2010 to September 2023. Although the models considered different approaches to volatility, the three volatility fits were similar. However, the FIGARCH and ARFIMA-GARCH fits produced the highest and lowest values, respectively, also considering that the fitted volatilities allowed us to identify some crisis periods; for example, the pandemic starting in March 2020.

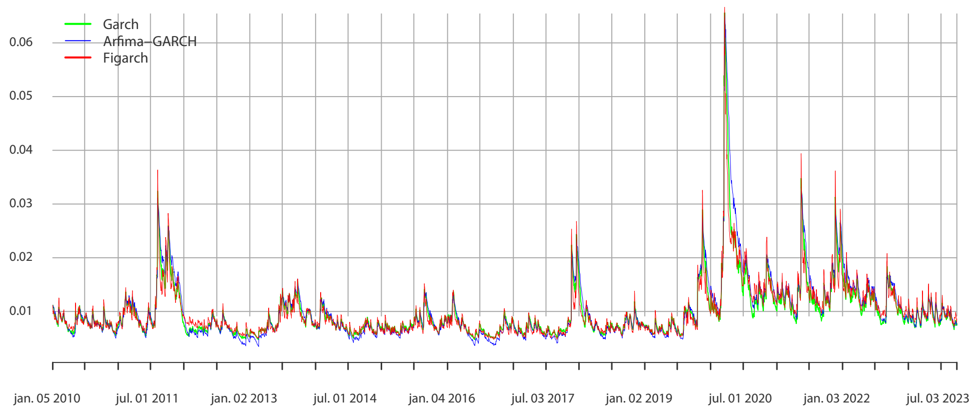

Figure 7. Estimated volatility under GARCH, ARFIMA-GARCH, and FIGARCH models for SSP log returns (January 2010–September 2023).

In other words, the selected models more effectively captured the intrinsic dispersion of the SSP index by showing greater relative volatility, not only during the period of the global health crisis compared to the estimates obtained from traditional autoregressive conditional heteroscedasticity models, but also during the post-crisis period. This greater volatility is consistent with the global fiscal imbalances that have arisen and their impact on the atypical returns or values of the risk premiums demanded by investors. In this way, it is formally observed that as the returns on financial assets $\mathbb{E}_t[r_{t+1} - r^f_{t+1}]$ become unstable, the correlation $Corr\left(u_\lambda(w_{t+1}), r_{t+1} - r^f_{t+1}\right)$ between the marginal benefit of investors' wealth and these returns decreases, which aggravates their volatility $Var(r_{t+1})$.

Therefore, the ability to identify the volatility of returns, especially in atypical fluctuations in the financial system, is essential for risk management and the optimization of investment portfolios. This allows investors to plan and mitigate the impacts of future events with similar characteristics.

6. Discussion and Conclusions

In this study, the GARCH, ARFIMA-GARCH, and FIGARCH models were considered to model the SSP log return time series from January 2010 to September 2023. The models were selected in response to the high volatility of log returns. Palma (2016) explained that the financial time series and stock indices presented generally log memory in their processes, which motivated the use of fractionally integrated and heteroskedastic models for this study. In addition, model parameters were estimated, where for ARFIMA-GARCH (selected over its competitors based on AIC and BIC), the fractionally integrated parameter was not significant. However, the residuals of this model were white noise, confirmed by the weighted Ljung–Box test. For the GARCH and FIGARCH models, all parameters were significant, but their residuals were not white noise, likely because volatility rose in some periods due to crises (Figure 7). In conclusion, fractionally integrated and heteroskedastic models offer a more precise fit for SSP log return volatility compared to a classical GARCH model.

Some implications obtained based on the estimation of SSP log-returns given in this study are as follows:

1. Volatility estimation was implicit in all models because they considered the actual value of the SSP index. However, the SSP log return volatility of assets is influenced by structural market factors linked to investor preferences.
2. The proposed ARFIMA-GARCH model detects the influence of volatility increments of the SSP index linked to external factors that impact the economic outlook (Idrovo-Aguirre and Contreras-Reyes 2021b), such as China's economic slowdown in 2012 and the subprime crisis in 2008.
3. In addition, the volatility increments of the SSP index may be affected by local macroeconomic variables such as the monthly economic activity index[3] (Troncoso et al. 2023), which summarized economic activity in several sectors in a given month, as noted by Donders Canto (2015), who supports the idea that a systematic variable affecting the price structure or impacting company dividends also affects stock market returns.
4. Also, IMACEC volatility changed markedly during the pandemic (Idrovo-Aguirre and Contreras-Reyes 2021a) due to the fluctuations in its values; a result of the global pandemic's economic impact. In this sense, IMACEC volatility is related to the infectious disease volatility index (Romero-Meza et al. 2021), which contains data on infectious diseases and economic data from the United States and other countries.
5. In uncertain economic situations like these, saving capacity tends to fall, reducing the capital available for the stock market. Therefore, risk aversion intensifies, and liquidity shifts toward investments that are considered safe, such as fixed income, and away from equities. This movement contributed significantly to IMACEC volatility during the study period.

Beyond the proposed FIGARCH and ARFIMA-GARCH models, several other extensions could be considered, such as FIAPARCH and HYGARCH (Khumalo et al. 2023), EGARCH (Ghalanos 2023), or TGARCH (Chávez et al. 2023). Further work should consider multivariate time series by mixing SSP log returns with data from other stock markets, such as S&P500, NASDAQ, and those in other Latin American countries (Aragón 2017). For this approach, it could be useful to consider vector ARFIMA (Contreras-Reyes 2022), multivariate SETAR (Contreras-Reyes 2024), and multivariate GARCH (Bauwens et al. 2006) models. In addition, recurrent neuronal network methods could be implemented, given the high number of observations (Ubal et al. 2023).

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Notes

[1] Spanish acronym IPSA.

[2] Except for lags 10 and 21 for $|r_t|$, where sample ACF was higher than Bartlett's bands. However, the sample ACF scale was small in a range of -0.04–0.04, indicating that deviations were of a low magnitude.

[3] Spanish acronym IMACEC.

References

Alfaro, Rodrigo A., and Carmen Gloria Silva. 2008. Volatilidad de índices accionarios: El caso del IPSA. *Cuadernos de Economía* 45: 217–33. [CrossRef]

Andersen, Torben G., and Timo Teräsvirta. 2009. Realized volatility. In *Handbook of Financial Time Series*. Berlin and Heidelberg: Springer, pp. 555–75.

Aragón, Martha Beatriz Mota. 2017. Volatilidad del mercado integrado latinoamericano: Un enfoque multivariado. *Estocástica: Finanzas y Riesgo* 7: 9–26.

Baillie, Richard T., Ching-Fan Chung, and Margie A. Tieslau. 1996a. Analysing inflation by the fractionally integrated ARFIMA–GARCH model. *Journal of Applied Econometrics* 11: 23–40. [CrossRef]

Baillie, Richard T., Tim Bollerslev, and Hans Ole Mikkelsen. 1996b. Fractionally integrated generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 74: 3–30. [CrossRef]

Bauwens, Luc, Sébastien Laurent, and Jeroen VK Rombouts. 2006. Multivariate GARCH models: A survey. *Journal of Applied Econometrics* 21: 79–109. [CrossRef]

Bollerslev, Tim. 1986. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 31: 307–27. [CrossRef]

Bollerslev, Tim. 1987. A conditionally heteroskedastic time series model for speculative prices and rates of return. *The Review of Economics and Statistics* 69: 542–47. [CrossRef]

Bollerslev, Tim, and Hans Ole Mikkelsen. 1996. Modeling and pricing long memory in stock market volatility. *Journal of Econometrics* 73: 151–84. [CrossRef]

Box, George E. P., Gwilym M. Jenkins, Gregory C. Reinsel, and Greta M. Ljung. 2015. *Time Series Analysis: Forecasting and Control*. Hoboken: John Wiley & Sons.

Campbell, John Y., Andrew W. Lo, A. Craig MacKinlay, and Robert F. Whitelaw. 1998. The econometrics of financial markets. *Macroeconomic Dynamics* 2: 559–62. [CrossRef]

Chávez, Diego, Javier E. Contreras-Reyes, and Byron J. Idrovo-Aguirre. 2023. A Threshold GARCH Model for Chilean Economic Uncertainty. *Journal of Risk and Financial Management* 16: 20. [CrossRef]

Cheung, Yin-Wong, and Kon S. Lai. 1995. Lag order and critical values of the augmented Dickey–Fuller test. *Journal of Business & Economic Statistics* 13: 277–80.

Contreras-Reyes, Javier E. 2022. Rényi entropy and divergence for VARFIMA processes based on characteristic and impulse response functions. *Chaos, Solitons & Fractals* 160: 112268.

Contreras-Reyes, Javier E. 2024. Information quantity evaluation of multivariate SETAR processes of order one and applications. *Statistical Papers* 65: 1553–1573. [CrossRef]

Contreras-Reyes, Javier E., and Wilfredo Palma. 2013. Statistical analysis of autoregressive fractionally integrated moving average models in R. *Computational Statistics* 28: 2309–31. [CrossRef]

Ding, Zhuanxin, Clive W. J. Granger, and Robert F. Engle. 1993. A Long Memory Property of Stock Market Returns and a New Model. *Journal of Empirical Finance* 1: 83–106. [CrossRef]

Donders Canto, Pablo. 2015. Determinantes Comunes Entre el IMACEC y el IPSA: Un Enfoque de Largo Plazo en Base a la Teorá de la Cointegración. Bachelor's thesis, Universidad de Chile, Santiago, Chile, January. Available online: https://repositorio.uchile.cl/handle/2250/129696 (accessed on 1 April 2023).

Engle, Robert F. 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50: 987–1007. [CrossRef]

Fisher, Thomas J., and Colin M. Gallagher. 2012. New weighted Portmanteau statistics for time series goodness of fit testing. *Journal of the American Statistical Association* 107: 777–87. [CrossRef]

Ghalanos, Alexios. 2023. Introduction to the Rugarch Package (Version 1.4-3). Available online: https://cran.r-project.org/web/packages/rugarch/vignettes/Introduction_to_the_rugarch_package.pdf (accessed on 1 July 2023).

Gil-Alana, Luis A. 2006. Fractional integration in daily stock market indexes. *Review of Financial Economics* 15: 28–48. [CrossRef]

Gonzalez-Rivera, Gloria, Tae-Hwy Lee, and Santosh Mishra. 2004. Forecasting volatility: A reality check based on option pricing, utility function, value-at-risk, and predictive likelihood. *International Journal of Forecasting* 20: 629–45. [CrossRef]

Granger, Clive WJ, and Roselyne Joyeux. 1980. Introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis* 1: 15–29. [CrossRef]

Hosking, Jonathan. 1981. Fractional differencing. *Biometrika* 68: 165–76. [CrossRef]

Idrovo-Aguirre, Byron J., and Javier E. Contreras-Reyes. 2021a. Monetary fiscal contributions to households and pension fund withdrawals during the COVID-19 pandemic: An approximation of their impact on construction labor supply in Chile. *Social Sciences* 10: 417. [CrossRef]

Idrovo-Aguirre, Byron J., and Javier E. Contreras-Reyes. 2021b. The response of housing construction to a copper price shock in Chile (2009–2020). *Economies* 9: 98. [CrossRef]

Khumalo, Moses, Hopolang Mashele, and Modisane Seitshiro. 2023. Quantification of the stock market value at risk by using FIAPARCH, HYGARCH and FIGARCH models. *Data Science in Finance and Economics* 3: 380–400. [CrossRef]

Nelson, Daniel B. 1991. Conditional Heteroskedasticity in Asset Returns: A New Approach. *Econometrica* 59: 347–70. [CrossRef]

Palma, Wilfredo. 2007. *Long-Memory Time Series: Theory and Methods*. Hoboken: John Wiley & Sons.

Palma, Wilfredo. 2016. *Time Series Analysis.* Hoboken: John Wiley & Sons.

R Core Team. 2023. A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. Available online: http://www.R-project.org (accessed on 1 July 2023).

Ramírez-Parietti, Ignacio, Javier E. Contreras-Reyes, and Byron J. Idrovo-Aguirre. 2021. Cross-sample entropy estimation for time series analysis: A nonparametric approach. *Nonlinear Dynamics* 105: 2485–508. [CrossRef]

Romero-Meza, Rafael, Semei Coronado, and Fabricio Ibañez-Veizaga. 2021. COVID-19 y causalidad en la volatilidad del mercado accionario chileno. *Estudios Gerenciales* 37: 242–50. [CrossRef]

Troncoso, Nicolás, Javier E. Contreras-Reyes, and Byron J. Idrovo-Aguirre. 2023. Measuring economic uncertainty synchrony with cross-sample entropy under common external factors: The case of Chile. *Fluctuation and Noise Letters* 22: 2350046. [CrossRef]

Tsay, Ruey S. 2005. *Analysis of Financial Time Series*. Hoboken: John Wiley & Sons.

Ubal, Cristian, Gustavo Di-Giorgi, Javier E. Contreras-Reyes, and Rodrigo Salas. 2023. Predicting the long-term dependencies in time series using recurrent artificial neural networks. *Machine Learning and Knowledge Extraction* 5: 1340–58. [CrossRef]